

Sergey Zeltyn

February 2005

zeltyn@ie.technion.ac.il

**STAT 991. Service Engineering.
The Wharton School. University of Pennsylvania.**

**Abandonment and Customers' Patience
in Tele-Queues.
The Palm/Erlang-A Model.**

Based on:

- Mandelbaum A. and Zeltyn S.
The Palm/Erlang-A Queue, with Applications to Call Centers.
Lecture note to *Service Engineering* course.
http://iew3.technion.ac.il/serveng/References/Erlang_A_Dec04.pdf
- Mandelbaum A. *Service Engineering* course, Technion.
<http://iew3.technion.ac.il/serveng2005W>

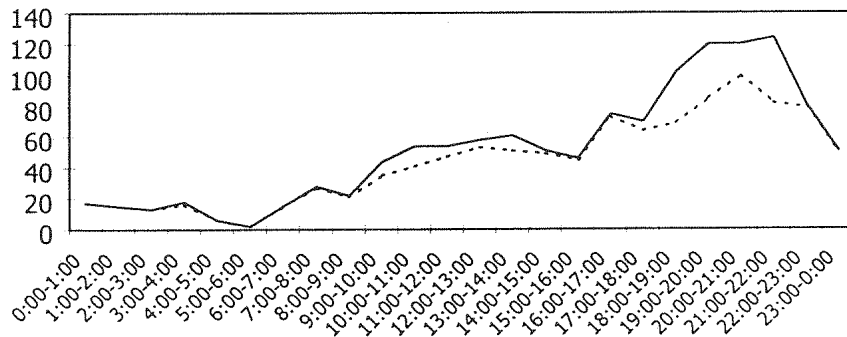
No abandonment in models of the previous lecture.

However, abandonment takes place and can be very significant and very important.

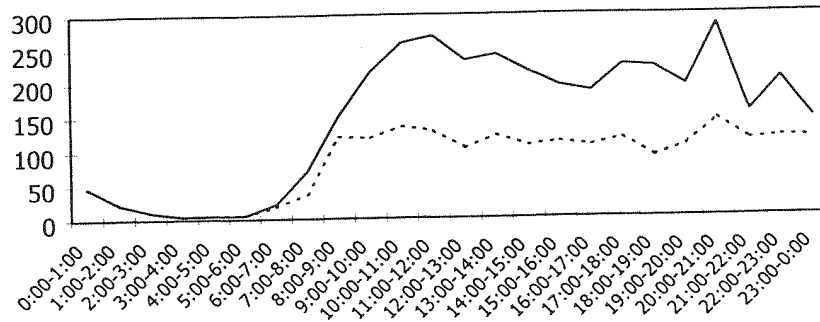
Example 1. “Catastrophic situation”.

Call center of telephone company.

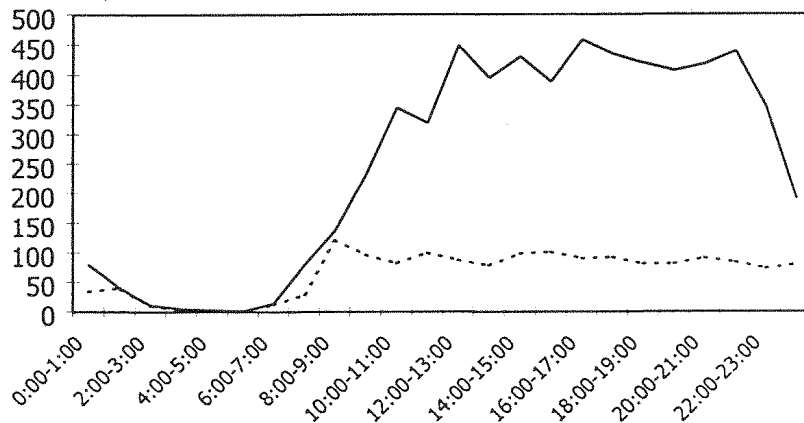
Average wait 72 sec, 81% calls answered (Saturday, 06/11/99)



Average wait 217 sec, 53% calls answered (Thursday, 25/11/99)



Average wait 376 sec, 24% calls answered (Sunday, 21/11/99)



Example 2. Moderate abandonment.

(Moderate, but still very important.)

Health Insurance. Charlotte – Center. ACD report.

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
Total	20,577	19,860	3.5%	30	307	95.1%	
8:00	332	308	7.2%	27	302	87.1%	59.3
8:30	653	615	5.8%	58	293	96.1%	104.1
9:00	866	796	8.1%	63	308	97.1%	140.4
9:30	1,152	1,138	1.2%	28	303	90.8%	211.1
10:00	1,330	1,286	3.3%	22	307	98.4%	223.1
10:30	1,364	1,338	1.9%	33	296	99.0%	222.5
11:00	1,380	1,280	7.2%	34	306	98.2%	222.0
11:30	1,272	1,247	2.0%	44	298	94.6%	218.0
12:00	1,179	1,177	0.2%	1	306	91.6%	218.3
12:30	1,174	1,160	1.2%	10	302	95.5%	203.8
13:00	1,018	999	1.9%	9	314	95.4%	182.9
13:30	1,061	961	9.4%	67	306	100.0%	163.4
14:00	1,173	1,082	7.8%	78	313	99.5%	188.9
14:30	1,212	1,179	2.7%	23	304	96.6%	206.1
15:00	1,137	1,122	1.3%	15	320	96.9%	205.8
15:30	1,169	1,137	2.7%	17	311	97.1%	202.2
16:00	1,107	1,059	4.3%	46	315	99.2%	187.1
16:30	914	892	2.4%	22	307	95.2%	160.0
17:00	615	615	0.0%	2	328	83.0%	135.0
17:30	420	420	0.0%	0	328	73.8%	103.5
18:00	49	49	0.0%	14	180	84.2%	5.8

Abandonment Important Practically

- One of two customer-subjective performance measures (2nd=Redials);
- Lost business (now);
- Poor service level (future losses);
- 1-800 costs (out-of-pocket vs. alternative);
- Self-selection: the “fittest survive” and wait less;
- Must account for (carefully) in models and measures.
Otherwise, wrong picture of reality: misleading performance measures, hence staffing.
- Unstable models (vs. robustness).

Abandonment also Interesting Theoretically

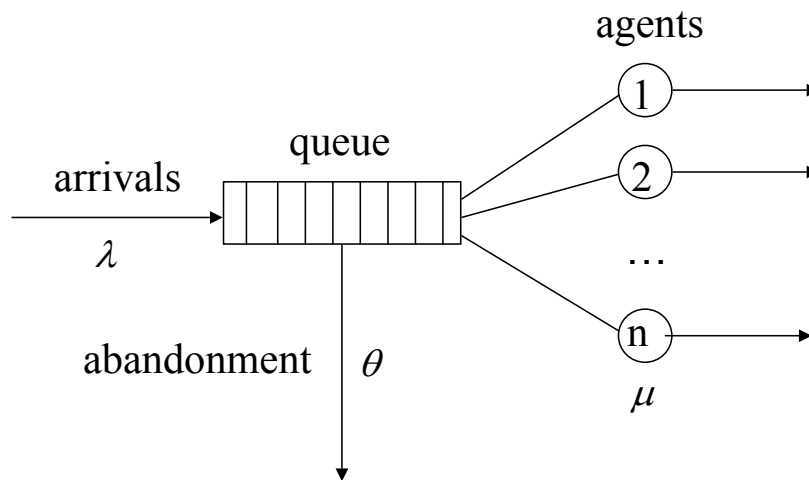
- Queueing Science
(Paradigm: experiment, measure, model, validate);
- Research: OR + Psychology + Marketing
(Modelling: steady-state, transient, equilibrium);
- Wide Scope of Applications: in addition to Phone,
 - VRU/IVR: opt-out-rates;
 - Internet: business-drivers (60% and more).

The Erlang-A (Palm, M/M/n+M) Model

Simplest model with abandonment, used by well-run call centers.

Building blocks:

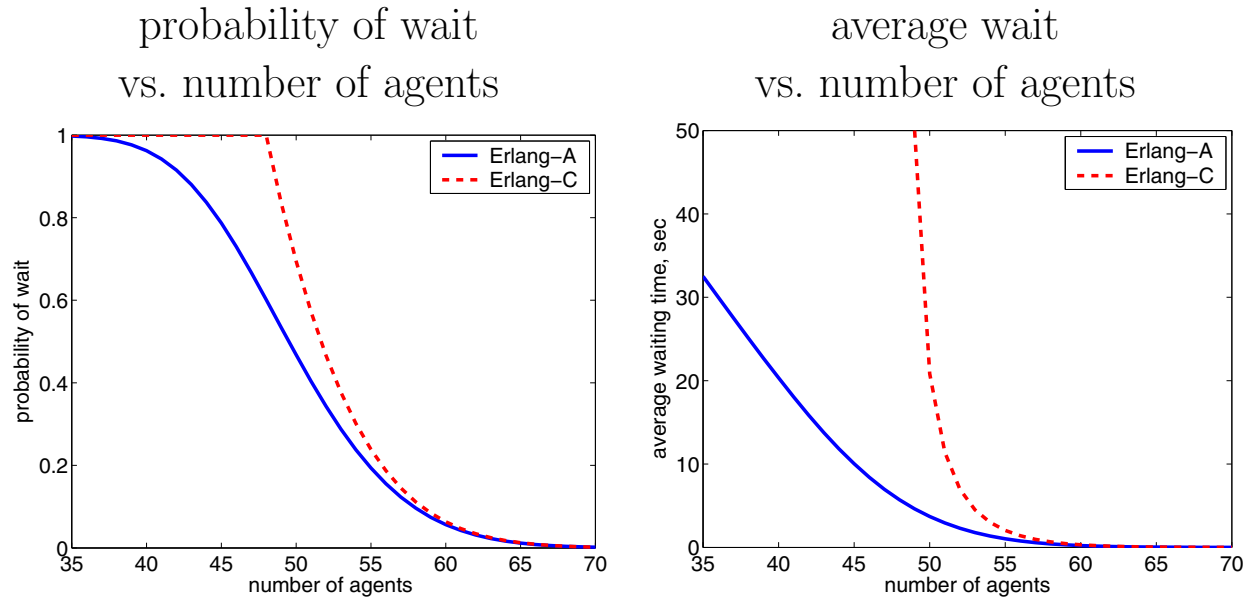
- λ – Poisson arrival rate.
- μ – Exponential service rate.
- n – number of service agents.
- θ – individual abandonment rate.



- **Patience time** $\tau \sim \exp(\theta)$:
time a customer is willing to wait for service;
- **Offered wait** V :
waiting time of a customer with infinite patience;
- If $\tau \leq V$, customer abandons; otherwise, gets service;
- **Actual wait** $W_q = \min(\tau, V)$.

Erlang-A vs. Erlang-C

48 calls per min, 1 min average service time,
2 min average patience



If 50 agents:

	M/M/n	M/M/n+M	M/M/n, $\lambda \downarrow 3.1\%$
Fraction abandoning	—	3.1%	—
Average waiting time	20.8 sec	3.7 sec	8.8 sec
Waiting time's 90-th percentile	58.1 sec	12.5 sec	28.2 sec
Average queue length	17	3	7
Agents' utilization	96%	93%	93%

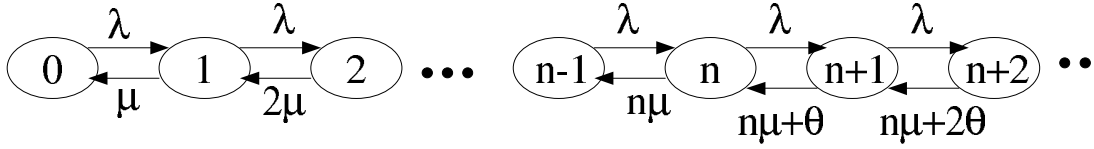
“The fittest survive” and wait less - much less.

Abandonment reduces workload when needed – at high-congestion periods.

Erlang-A: Birth-and-Death Process

$L(t)$ – number-in-system at time t (served plus queued);
 $L = \{L(t), t \geq 0\}$ – Markov birth-and-death process.

Transition-rate diagram



Steady-state equations:

$$\begin{cases} \lambda\pi_j = (j+1) \cdot \mu\pi_{j+1}, & 0 \leq j \leq n-1 \\ \lambda\pi_j = (n\mu + (j+1-n)\theta) \cdot \pi_{j+1}, & j \geq n. \end{cases}$$

Steady-state distribution:

$$\pi_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!} \pi_0, & 0 \leq j \leq n \\ \prod_{k=n+1}^j \left(\frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \pi_0, & j \geq n+1, \end{cases}$$

where

$$\pi_0 = \left[\sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!} + \sum_{j=n+1}^{\infty} \prod_{k=n+1}^j \left(\frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \right]^{-1}.$$

Numerical drawback: infinite sums.

Stability

Erlang-A is always stable!

d_j – death-rate in state j , $0 < j < \infty$:

$$j \cdot \min(\mu, \theta) \leq d_j \leq j \cdot \max(\mu, \theta) .$$

Bounds are death rates of M/M/ ∞ queues with service rates $\min(\mu, \theta)$ and $\max(\mu, \theta)$.

Proof of stability:

$$\begin{aligned} \pi_0^{-1} &= \sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!} + \sum_{j=n+1}^{\infty} \prod_{k=n+1}^j \left(\frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \\ &\leq \sum_{j=0}^{\infty} \frac{(\lambda/\min(\mu, \theta))^j}{j!} = e^{-\lambda/\min(\mu, \theta)} . \end{aligned}$$

(Use that $n\mu + (k-n)\theta \geq k \min(\mu, \theta)$.)

Steady-state distribution via special functions (Palm):

Gamma function:

$$\Gamma(x) \triangleq \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

Incomplete Gamma function:

$$\gamma(x, y) \triangleq \int_0^y t^{x-1} e^{-t} dt, \quad x > 0, y \geq 0.$$

$$A(x, y) \triangleq \frac{x e^y}{y^x} \cdot \gamma(x, y) = 1 + \sum_{j=1}^{\infty} \frac{y^j}{\prod_{k=1}^j (x + k)}, \quad x > 0, y \geq 0.$$

Recall $E_{1,n}$ – *blocking probability* in M/M/n/n (Erlang-B):

$$E_{1,n} = \frac{\frac{(\lambda/\mu)^n}{n!}}{\sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!}} = \frac{(\lambda/\mu)^n}{e^{\lambda/\mu}} \cdot \frac{1}{\Gamma(n+1) - \gamma(n+1, \lambda/\mu)}.$$

Can be calculated also via recursion.

Then one can show:

$$\pi_j = \begin{cases} \pi_n \cdot \frac{n!}{j! \cdot \left(\frac{\lambda}{\mu}\right)^{n-j}}, & 0 \leq j \leq n, \\ \pi_n \cdot \frac{\left(\frac{\lambda}{\theta}\right)^{j-n}}{\prod_{k=1}^{j-n} \left(\frac{n\mu}{\theta} + k\right)}, & j \geq n+1, \end{cases}$$

where

$$\pi_n = \frac{E_{1,n}}{1 + \left[A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) - 1 \right] \cdot E_{1,n}}.$$

Operational Performance Measures

The most popular performance measure is $P\{W_q \leq T; \text{Sr}\}$ (or even worse $P\{W_q \leq T \mid \text{Sr}\}$).

We recommend:

- $P\{W_q \leq T; \text{Sr}\}$ - fraction of well-served;
- $P\{\text{Ab}\}$ - fraction of poorly-served.

or a four-dimensional refinement:

- $P\{W_q \leq T; \text{Sr}\}$ - fraction of well-served;
- $P\{W_q > T; \text{Sr}\}$ - fraction of served, with potential for improvement (say, a higher priority on next visit);
- $P\{W_q > \epsilon; \text{Ab}\}$ - fraction of poorly-served;
- $P\{W_q \leq \epsilon; \text{Ab}\}$ - fraction of those whose service-level is undetermined.

Properties of $P\{\text{Ab}\}$:

- $P\{\text{Ab}\}$ increases monotonically in θ, λ ;
 $P\{\text{Ab}\}$ decreases monotonically in n, μ
(Bhattacharya and Ephremides (1991))
- M/M/ n +G: if $E[\tau]$ is fixed, deterministic patience minimizes $P\{\text{Ab}\}$ (Mandelbaum and Zeltyn (2004))

Additional important performance measures:

- Delay probability $P\{W_q > 0\}$;
- Average wait $E[W_q]$;
- ASA (Average Speed of Answer) – used extensively in call centers; usually defined as $E[W_q | \text{Sr}]$;
- Agents' occupancy $\rho = \frac{\lambda \cdot (1 - P\{\text{Ab}\})}{n\mu}$.
- Average queue-length $E[L_q]$.

Operational Performance Measures: calculation via 4CallCenters

Performance measures of the form $E[f(V, \tau)]$.

Calculable, in numerically stable procedures.

For example,

$f(v, \tau)$	$E[f(V, \tau)]$
$1_{\{v > \tau\}}$	$P\{V > \tau\} = P\{\text{Ab}\}$
$1_{(t, \infty)}(v \wedge \tau)$	$P\{W_q > t\}$
$1_{(t, \infty)}(v \wedge \tau) 1_{\{v > \tau\}}$	$P\{W_q > t; \text{Ab}\}$
$(v \wedge \tau) 1_{\{v > \tau\}}$	$E\{W_q; \text{Ab}\}$
$g(v \wedge \tau)$	$E[g(W_q)]$

Operational Performance Measures: calculation via 4CallCenters

4CallCenters v2.01
File Table Settings Help

Performance Profiler | Staffing Query | Advanced Profiling | Advanced Queries | What-if Analysis

Performance Profiler Performance Profiler allows you to determine and optimize the Performance Level of your Call Center. Enter your call center's parameters below, then press 'Compute'.

Your Call Center's Parameters

- Number of Agents Answering Calls: 10
- Average Time to Handle One Call (mm:ss): 02:00
- Calls 60 minute: 300
- Average Callers' Patience (mm:ss): 02:00

Settings

- Features: Abandons
- Basic Interval: 60 minutes
- Target Time: 00:10 (mm:ss)

Change Settings

Compute | Add to Table | Delete Rows | Clear All | Export | Graph

	Target Time to Answer	Number of Agents	Average Handling Time	Calls per Interval	Average Patience	%Answer	%Abandon	%Answer within Target	%Abandon within Target
Results	00:10.0	10.0	02:00.0	300.0	02:00.0	87.5%	12.5%	55.7%	3.9%
1	00:30.0	10.0	02:00.0	300.0	02:00.0	87.5%	12.5%	71.1%	8.8%
2	00:00.0	10.0	02:00.0	300.0	02:00.0	87.5%	12.5%	45.8%	.0%
3									
4									
5									
6									
-									

Ready 28/12/2004 14:50

Start | Tera Term - ietw... | WinEdit - [C:\S... | Adobe Acrobat ... | 4CallCenters v... | 14:50

Erlang-A parameters:

$\lambda = 300$ calls per hour, $1/\mu = 2$ min, $n = 10$, $1/\theta = 2$ min.

Target times $T = 30$ sec, $\epsilon = 10$ sec.

- $P\{W_q \leq T; Sr\} = 71.1\%$;
- $P\{W_q > T; Sr\} = 87.5\% - 71.1\% = 16.4\%$;
- $P\{W_q > \epsilon; Ab\} = 12.5\% - 3.9\% = 8.6\%$;
- $P\{W_q \leq \epsilon; Ab\} = 3.9\%$.
- Delay probability $P\{W_q > 0\} = 100\% - 45.8\% = 54.2\%$.

Additional performance measures

4CallCenters v2.01

File Table Settings Help

Performance Profiler | Staffing Query | Advanced Profiling | Advanced Queries | What-if Analysis

Performance Profiler Performance Profiler allows you to determine and optimize the Performance Level of your Call Center. Enter your call center's parameters below, then press 'Compute'.

Your Call Center's Parameters

- ◆ Number of Agents Answering Calls: 10
- ◆ Average Time to Handle One Call (mm:ss): 02:00
- ◆ Calls 60 minute: 300
- ◆ Average Callers' Patience (mm:ss): 02:00

Settings

- ◆ **Features:** Abandons
- ◆ **Basic Interval:** 60 minutes
- ◆ **Target Time:** 00:10 (mm:ss)

Change Settings

Compute ◆ Add to Table Delete Rows Clear All Export Graph

	Agent's Occupancy	Agent's Availability	%Answer	%Abandon	Average Speed of Answer	Average Time in Queue	%Answer within Target	%Abandon within Target	Average Queue Length
Results	87.5%	12.5%	87.5%	12.5%	00:13.8	00:15.0	55.7%	3.9%	1.3
1									
2									
3									
4									
5									
6									
-									

Ready 28/12/2004 14:57

Start Tera Te... WinEdt... Adobe ... 4CallCe... fourCC... Docume... 14:57

- Average Time in Queue = $E[W_q] = 15$ sec;
- ASA = $E[W_q|Sr] = 13.8$ sec;
- Agents' Occupancy $\rho = 87.5\%$;
- Average Queue Length $E[L_q] = 1.3$.

Operational Performance Measures: calculation via special functions

For example,

$$\begin{aligned}P\{W_q > 0\} &= \sum_{j=n}^{\infty} \pi_j = \frac{A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) \cdot E_{1,n}}{1 + \left(A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) - 1\right) \cdot E_{1,n}}, \\P[\text{Ab}|W_q > 0] &= \frac{1}{\rho A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right)} + 1 - \frac{1}{\rho}, \\E[W_q|W_q > 0] &= \frac{1}{\theta} \cdot \left[\frac{1}{\rho A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right)} + 1 - \frac{1}{\rho} \right].\end{aligned}$$

Operational Performance Measures: calculation via M/M/n+G formulae

M/M/n+G – generalization of Erlang-A, patience times distributed with cdf $G(\cdot)$. See

<http://iew3.technion.ac.il/serveng/References/references.html>

- Mandelbaum A. and Zeltyn S. (2004) M/M/n+G queue. Summary of performance measures;
- Zeltyn S. (2004) Call centers with impatient customers: exact analysis and many-server asymptotics of the M/M/n+G queue, Ph.D. Thesis.

Explained how to adapt M/M/n+G to Erlang-A:

$$G(x) = 1 - e^{-\theta x}, \quad \theta > 0.$$

The relation $P\{Ab\}/E[W_q]$

Theoretical: In Erlang-A (and other queueing models with $\exp(\theta)$ patience):

$$P\{Ab\} = \theta \cdot E[W_q] .$$

Proof. Balance equation:

$$\theta \cdot E[L_q] = \lambda \cdot P\{Ab\} . \quad (1)$$

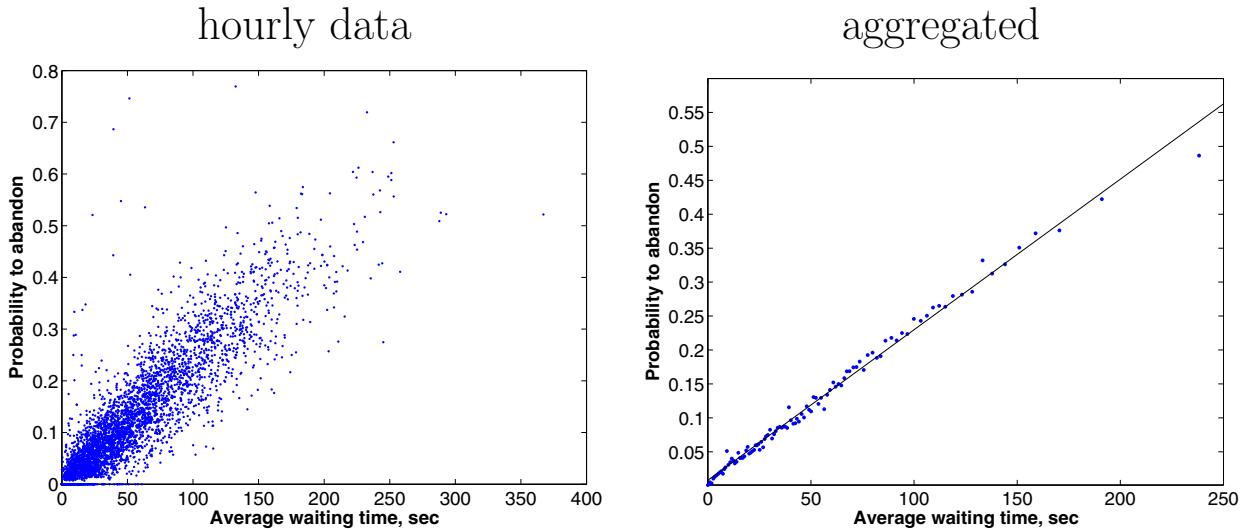
Little's formula:

$$E[L_q] = \lambda \cdot E[W_q] . \quad (2)$$

Substitute (2) into (1). ■

Empirical relations

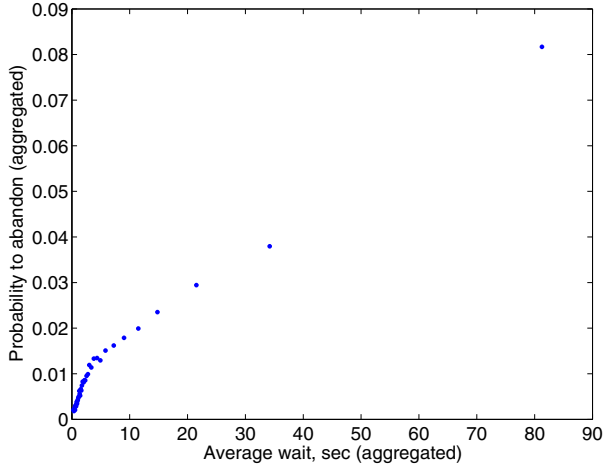
Israeli bank: yearly data



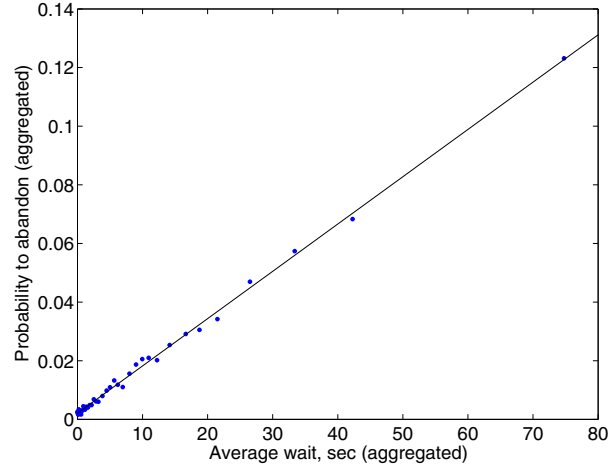
The graphs are based on 4158 hour intervals.

U.S. bank

Retail



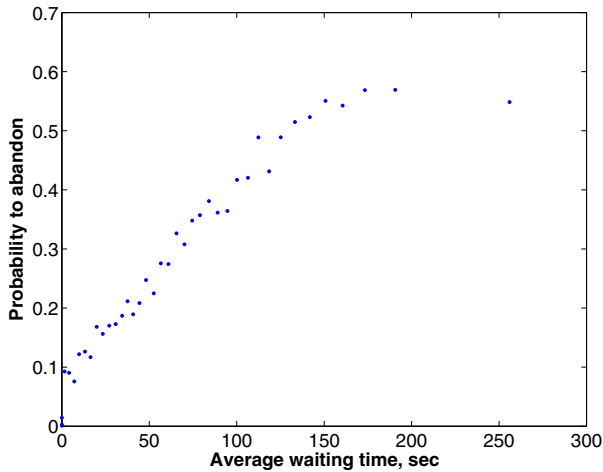
Telesales



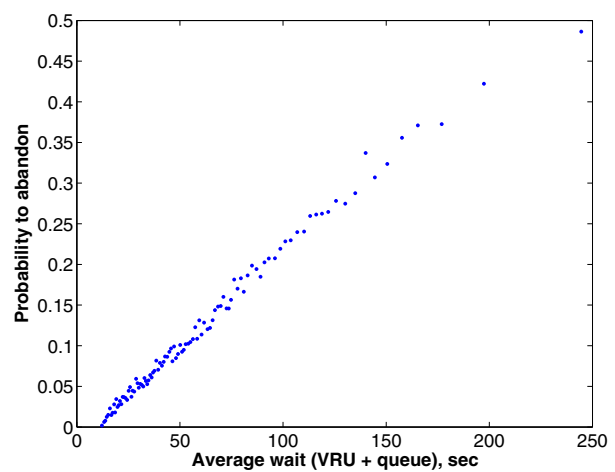
Retail – significant abandonment during first seconds of wait.

Linear patterns with non-zero intercepts

Israeli data: new customers



VRU-time included in wait



Left-hand plot \approx exp patience with balking:

0 with probability p , $\exp(\theta)$ with probability $(1 - p)$.

Right-hand plot \approx delayed patience: $c + \exp(\theta)$, $c > 0$.

Erlang-A: parameter estimation and prediction

Estimation: inference from historical data (e.g. exp, normal) were parameters assumed fixed over time.

Prediction: forecast behavior of sample outside of original data set.

Arrivals (λ)

- Typically Poisson, time-varying rates, constant at 15/30/60 min scale;
- Significant uncertainty concerning future rates \Rightarrow prediction;
- Predict separately *daily volumes* and *fraction* of arrivals per time interval.

Services (μ)

- Typically stable from day to day \Rightarrow estimation;
- Can change depending on time-of-day;
- Typically, service time \neq talk time.

First approach:

service time = talk time + wrap-up time (after-call work) + ...;

Second approach:

$$\text{service time} = \frac{\text{Total Working Time} - \text{Total Idle Time}}{\text{Number of Served Customers}}.$$

Number of agents (n)

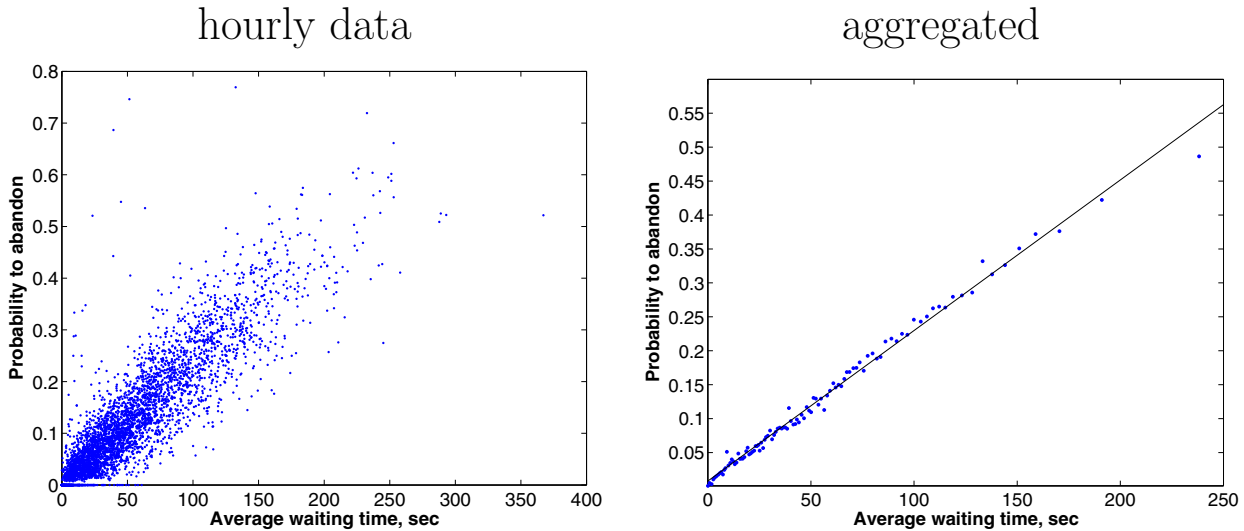
- Output of WFM software given λ , μ , θ , performance goals.
One gets number of FTE's (Full Time Equivalent positions).
- Agents on schedule = FTE's \cdot RSF (Rostered Staff Factor) (RSF > 1). Reasons: absenteeism, unscheduled breaks, ...
- Obtaining historical data on n can be hard.

Patience (θ)

Observations are **censored!** (heavily)

- Customer abandoned \Rightarrow patience τ known;
- Customer served \Rightarrow offered wait V known $\Rightarrow \tau > V$.

Avoiding direct “uncensoring”: use $P\{\text{Ab}\} = \theta \cdot E[W_q]$.



Regression \Rightarrow average patience $(1/\theta) \approx \frac{250}{0.56} \approx 446$ sec.

Estimating patience distribution

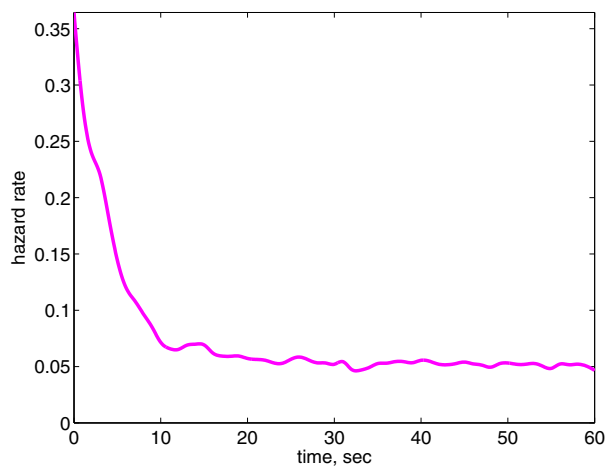
Are patience times really exponential?

To “uncensor data” use Kaplan-Meier (product-limit) estimator.

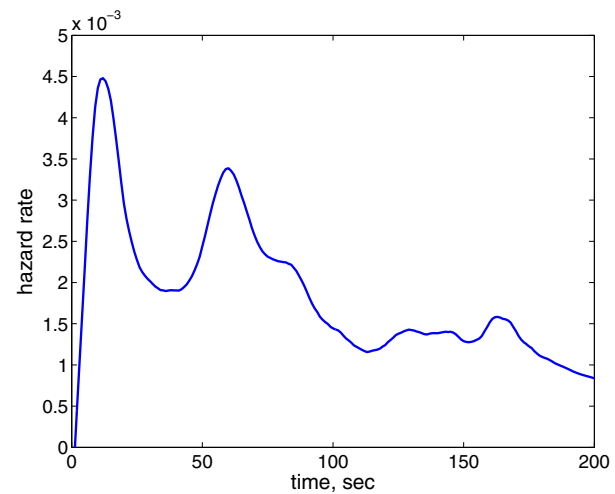
Output: estimates of survival function and hazard rate.

Empirical hazard rates of patience times

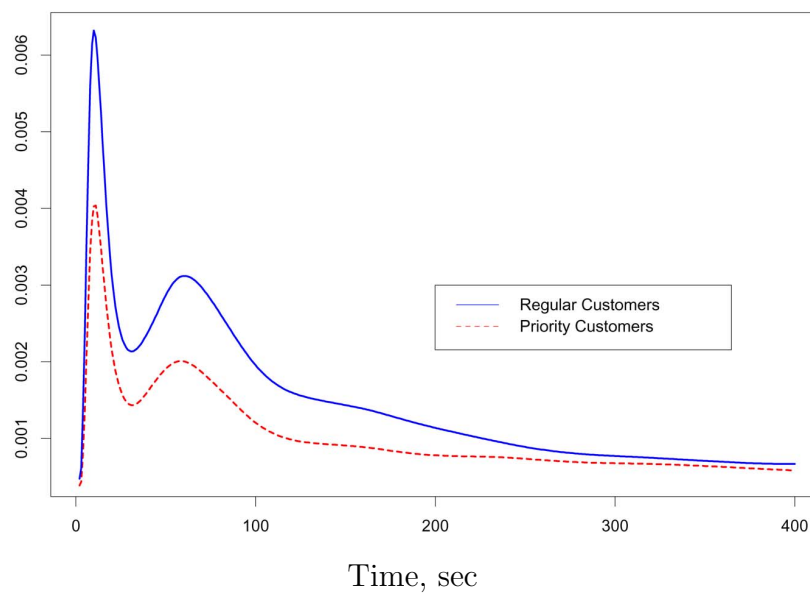
U.S. bank



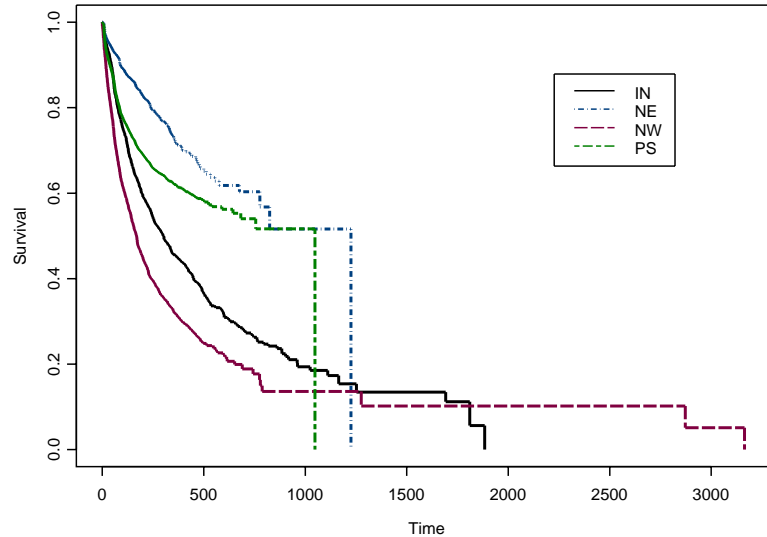
Israeli bank



Israeli bank: regular vs. priority customers



Israeli bank: service types



IN – Internet Assistance; NE – Stock Transactions;
NW – New Customers; PS – Regular

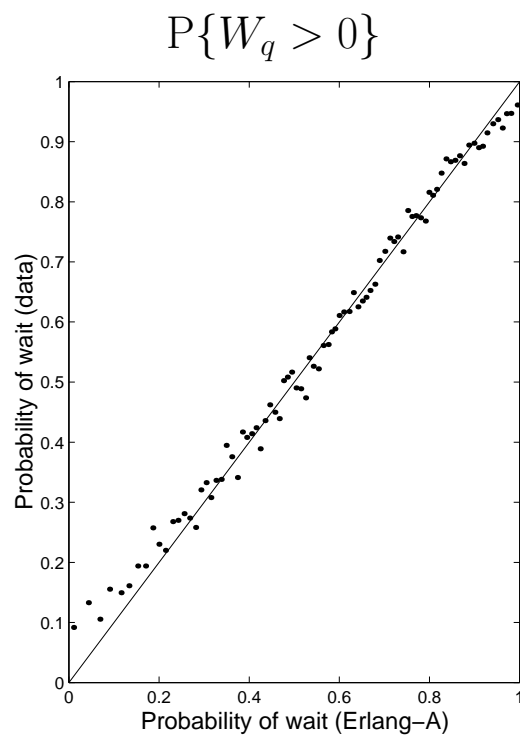
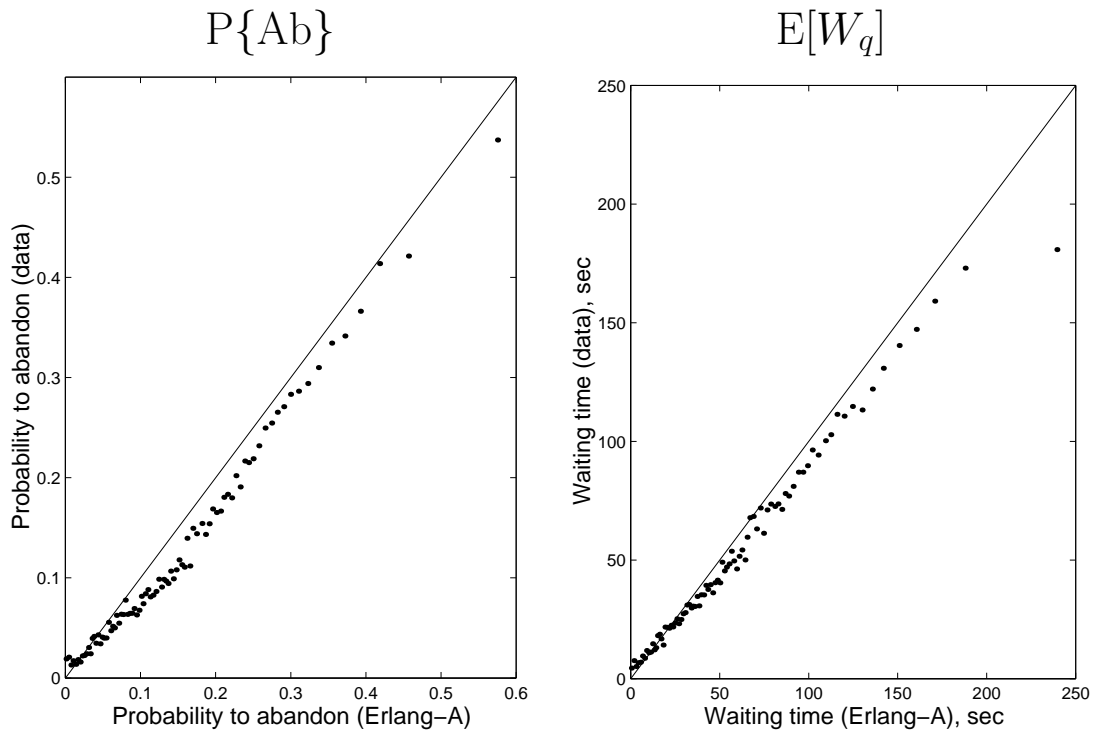
Conclusions:

- Patience times are, in general, non-exponential;
- Most tele-customers are **very** patient;
- Kaplan-Meier is very informative concerning patience *qualitative* patterns (abandonment peaks, comparisons, ...);
- Kaplan-Meier can be problematic concerning estimation of *quantitative* characteristics (mean, variance, median).
 $E[\tau] = \int_0^\infty \widehat{S}(x) dx$, where $S(x)$ - survival function of patience.
 However, $\widehat{S}(x)$ not reliable for large x .

Question: can we apply Erlang-A with non-exponential patience?

Fitting a simple model to a complex reality

Erlang-A Formulae vs. Data Averages (Israeli Bank)



Conclusions:

- Points: hourly data vs. Erlang-A output;
- Formulae with continuous n used;
- Patience estimated via $P\{\text{Ab}\}/E[W_q]$ relation;
- Erlang-A estimates – close upper bounds.

Fitting a simple model to a complex reality: Patience index

How to define (im)patience?

$$\begin{aligned}\text{Theoretical Patience Index} &\triangleq \frac{\text{time willing to wait}}{\text{time required to wait}} \\ &= \frac{\text{average patience}}{\text{average offered wait}}.\end{aligned}\quad (3)$$

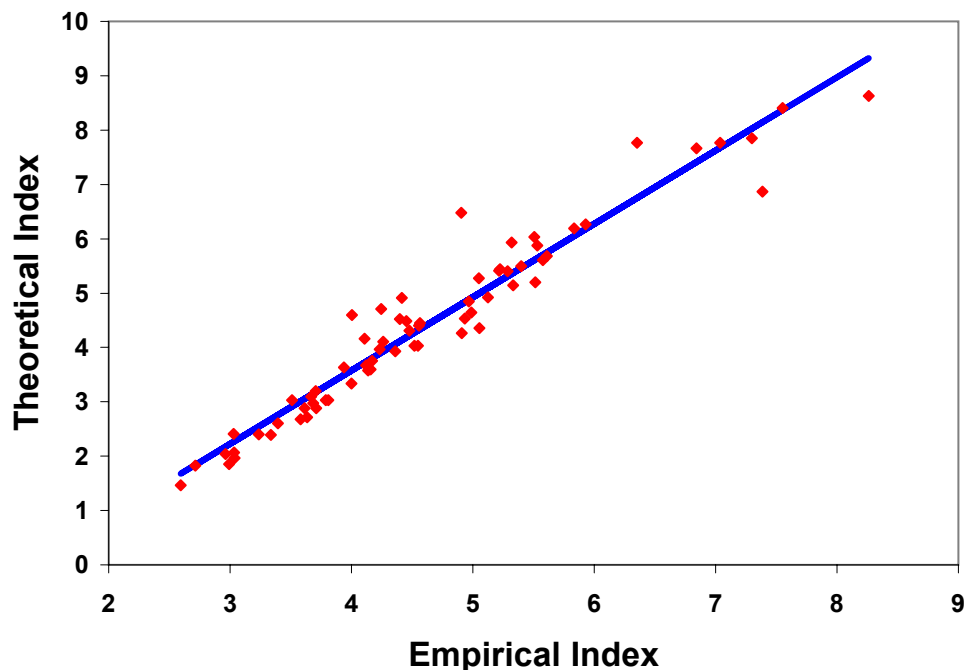
Calculation can be difficult.

$$\text{Empirical Patience Index} \triangleq \frac{\% \text{ served}}{\% \text{ abandoned}}.\quad (4)$$

Easily calculable from ACD reports.

If τ and V exponentially distributed, (4) is MLE of (3).

Patience index – empirical vs. theoretical



PATIENCE INDEX

- How to Define? Measure? Manage?

<u>Statistics</u>	<u>Time Till</u>	<u>Interpretation</u>
360K served (80%)	2 min.	? must = expect
90K abandon (20%)	1 min.	? willing to wait

“Time willing to wait” of served is **censored** by their “wait”.

“Uncensoring” (simplified)

Willing to wait $1 + 2 \times \frac{360K}{90K} = 1 + 2 \times 4 = 9 \text{ min.}$

Expect to wait $2 + 1 \times \frac{90K}{360K} = 2 + 1 \times \frac{1}{4} = 2.25 \text{ min.}$

Patience Index = $\frac{\text{time willing}}{\text{time expect}} = 4 = \frac{\# \text{ served/wait} > 0}{\# \text{ abandon/wait} > 0}$

\uparrow definition \uparrow measure

Customer-Focused Queueing Theory

Waiting experience can be summarized by:

1. Time that a customer *expects* to wait;
2. Time that a customer is *willing* to wait (τ , patience or need);
3. Time that a customer *must* wait (V , offered wait);
4. Time that a customer *actually* waits ($W_q = \min(\tau, V)$);
5. Time that a customer *perceives* waiting.

Experienced customers $\Rightarrow 1=3$;

Rational customers $\Rightarrow 4=5$;

Then left with (τ, V, W_q) , as introduced before.

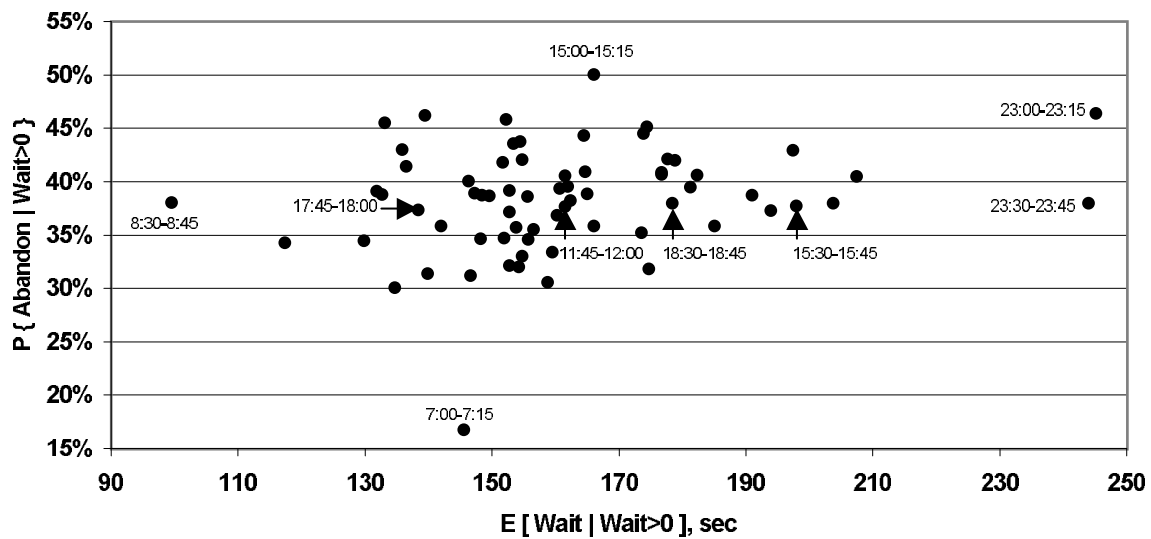
200 abandonment in Direct-Banking: perceived vs. actual waiting.

Reason to Abandon	Actual Abandon Time (sec)	Perceived Abandon Time (sec)	Perception Ratio
Fed up waiting (77%)	70	164	2.34
Not urgent (10%)	81	128	1.6
Forced to (4%)	31	35	1.1
Something came up (6%)	56	53	0.95
Expected call-back (3%)	13	25	1.9

Adaptive behavior of impatient customers

Question: Do customers adapt their patience to system performance (offered wait)?

Israeli bank: Internet-support customers



Rational abandonment from invisible queues: Mandelbaum, Shimkin, Zohar.

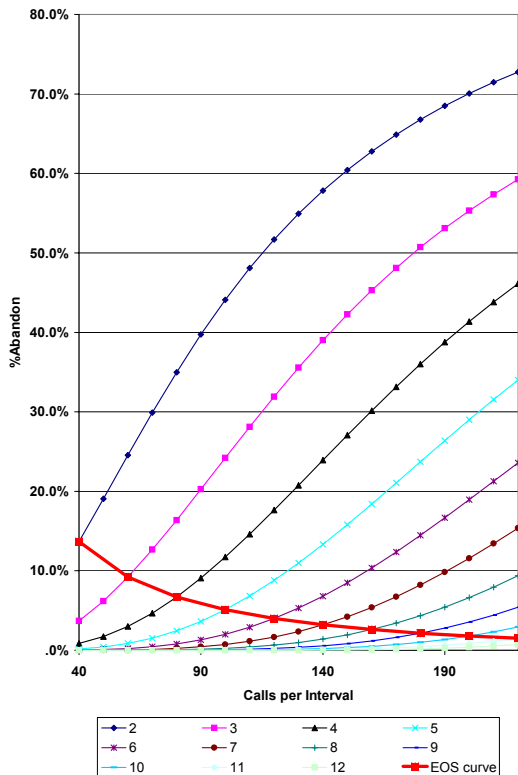
Advanced features of 4CallCenters

Advanced profiling

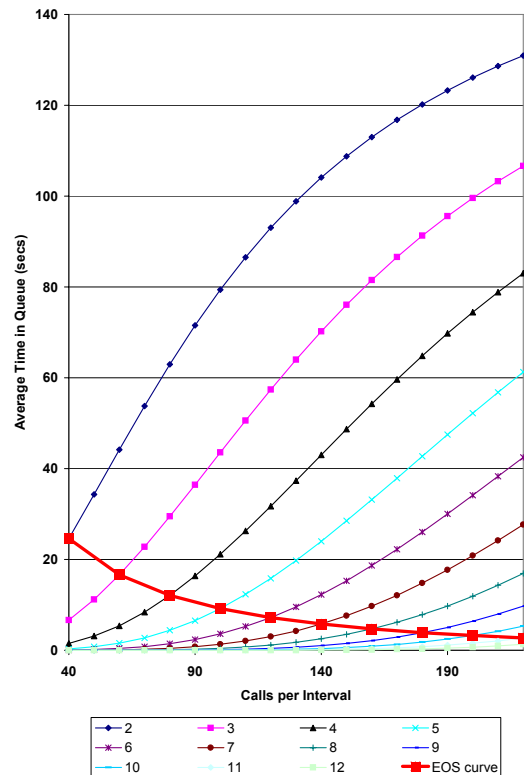
Vary input parameters of Erlang-A and display output (performance measures) in a table or graphically.

Example: $1/\mu = 2$ minutes, $1/\theta = 3$ minutes;
 λ varies from 40 to 230 calls per hour, in steps of 10;
 n varies from 2 to 12.

Probability to abandon



Average wait



Red curve: EOS (Economies-Of-Scale).

Why the two graphs are similar?

Advanced staffing queries

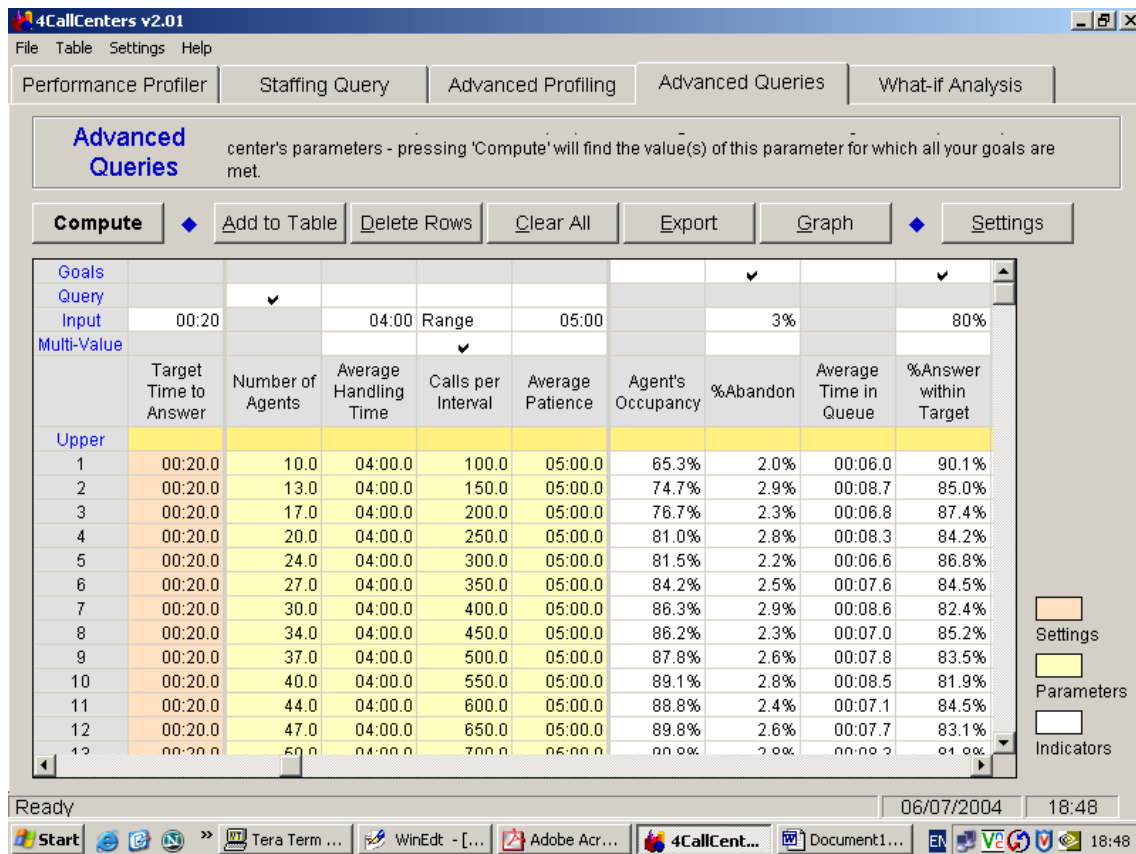
Multiple performance goals.

Example: $1/\mu = 4$ minutes, $1/\theta = 5$ minutes;
 λ varies from 100 to 1200, in steps of 50.

Performance targets:

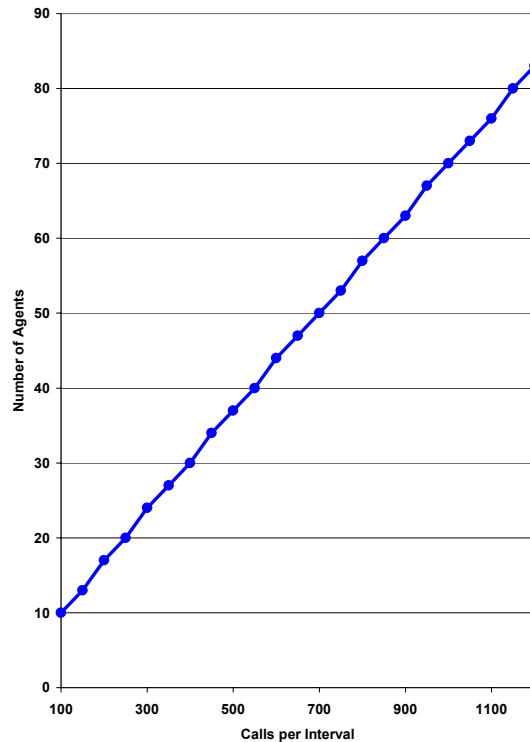
$$P\{Ab\} \leq 3\%; \quad P\{W_q < 20 \text{ sec}; Sr\} \geq 0.8.$$

4CallCenters output

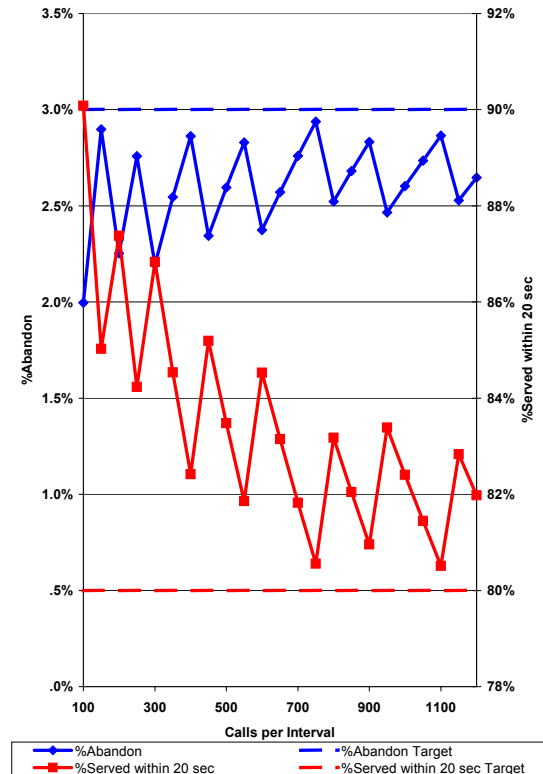


4CallCenters. Advanced staffing queries. Dynamics of staffing level and performance.

Recommended staffing level



Target performance measures



EOS: 10 agents needed for 100 calls per hour but only 83 for 1200 calls per hour.