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**STAT 991. Service Engineering.**  
**The Wharton School. University of Pennsylvania.**

**Abandonment and Customers' Patience  
in Tele-Queues.**  
**The Palm/Erlang-A Model.**

Based on:

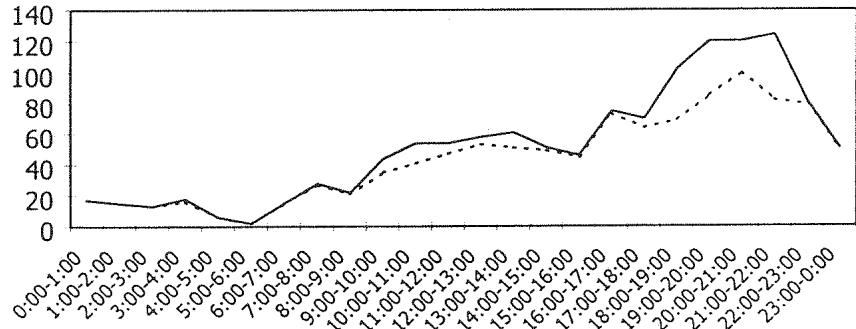
- Mandelbaum A. and Zeltyn S.  
The Palm/Erlang-A Queue, with Applications to Call Centers.  
Lecture note to *Service Engineering* course.  
[http://iew3.technion.ac.il/serveng/References/Erlang\\_A\\_Dec04.pdf](http://iew3.technion.ac.il/serveng/References/Erlang_A_Dec04.pdf)
- Mandelbaum A. *Service Engineering* course, Technion.  
<http://iew3.technion.ac.il/serveng2005W>

No abandonment in models of the previous lecture.

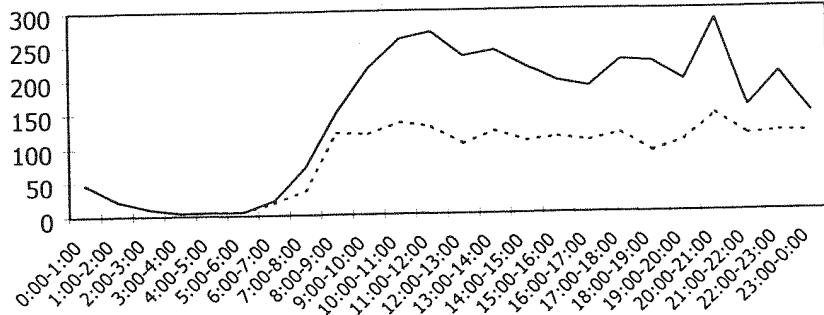
**However, abandonment takes place and can be very significant and very important.**

**Example 1. “Catastrophic situation”.**  
**Call center of telephone company.**

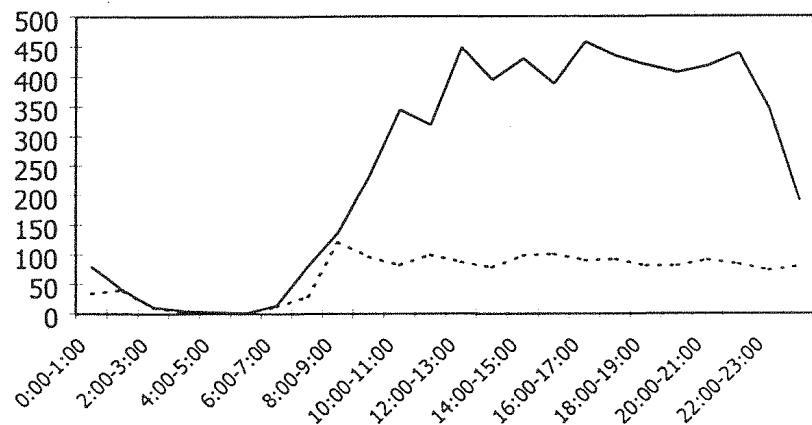
Average wait 72 sec, 81% calls answered (Saturday, 06/11/99)



Average wait 217 sec, 53% calls answered (Thursday, 25/11/99)



Average wait 376 sec, 24% calls answered (Sunday, 21/11/99)



## Example 2. Moderate abandonment.

(Moderate, but still very important.)

### Health Insurance. Charlotte – Center. ACD report.

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
Total	20,577	19,860	<b>3.5%</b>	<b>30</b>	307	95.1%	
8:00	332	308	7.2%	27	302	87.1%	59.3
8:30	653	615	5.8%	58	293	96.1%	104.1
9:00	866	796	8.1%	63	308	97.1%	140.4
9:30	1,152	1,138	1.2%	28	303	90.8%	211.1
10:00	1,330	1,286	3.3%	22	307	98.4%	223.1
10:30	1,364	1,338	1.9%	33	296	99.0%	222.5
11:00	1,380	1,280	7.2%	34	306	98.2%	222.0
11:30	1,272	1,247	2.0%	44	298	94.6%	218.0
12:00	1,179	1,177	0.2%	1	306	91.6%	218.3
12:30	1,174	1,160	1.2%	10	302	95.5%	203.8
13:00	1,018	999	1.9%	9	314	95.4%	182.9
<b>13:30</b>	<b>1,061</b>	<b>961</b>	<b>9.4%</b>	<b>67</b>	<b>306</b>	<b>100.0%</b>	<b>163.4</b>
14:00	1,173	1,082	7.8%	78	313	99.5%	188.9
<b>14:30</b>	<b>1,212</b>	<b>1,179</b>	<b>2.7%</b>	<b>23</b>	<b>304</b>	<b>96.6%</b>	<b>206.1</b>
15:00	1,137	1,122	1.3%	15	320	96.9%	205.8
15:30	1,169	1,137	2.7%	17	311	97.1%	202.2
16:00	1,107	1,059	4.3%	46	315	99.2%	187.1
16:30	914	892	2.4%	22	307	95.2%	160.0
<b>17:00</b>	<b>615</b>	<b>615</b>	<b>0.0%</b>	<b>2</b>	<b>328</b>	<b>83.0%</b>	<b>135.0</b>
17:30	420	420	0.0%	0	328	73.8%	103.5
18:00	49	49	0.0%	14	180	84.2%	5.8

## **Abandonment Important Practically**

- One of two customer-subjective performance measures (2<sup>nd</sup>=Redials);
- Lost business (now);
- Poor service level (future losses);
- 1-800 costs (out-of-pocket vs. alternative);
- Self-selection: the “fittest survive” and wait less;
- Must account for (carefully) in models and measures.  
Otherwise, wrong picture of reality: misleading performance measures, hence staffing.
- Unstable models (vs. robustness).

## **Abandonment also Interesting Theoretically**

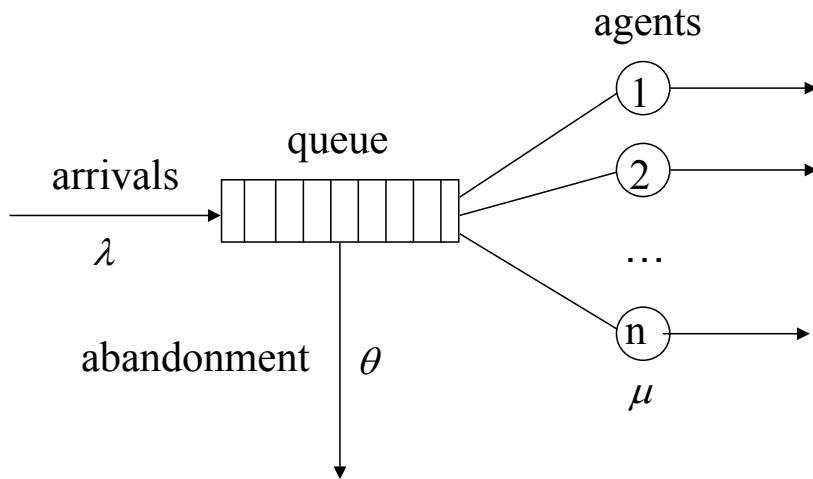
- Queueing Science  
(Paradigm: experiment, measure, model, validate);
- Research: OR + Psychology + Marketing  
(Modelling: steady-state, transient, equilibrium);
- Wide Scope of Applications: in addition to Phone,
  - VRU/IVR: opt-out-rates;
  - Internet: business-drivers (60% and more).

# The Erlang-A (Palm, M/M/n+M) Model

Simplest model with abandonment, used by well-run call centers.

## Building blocks:

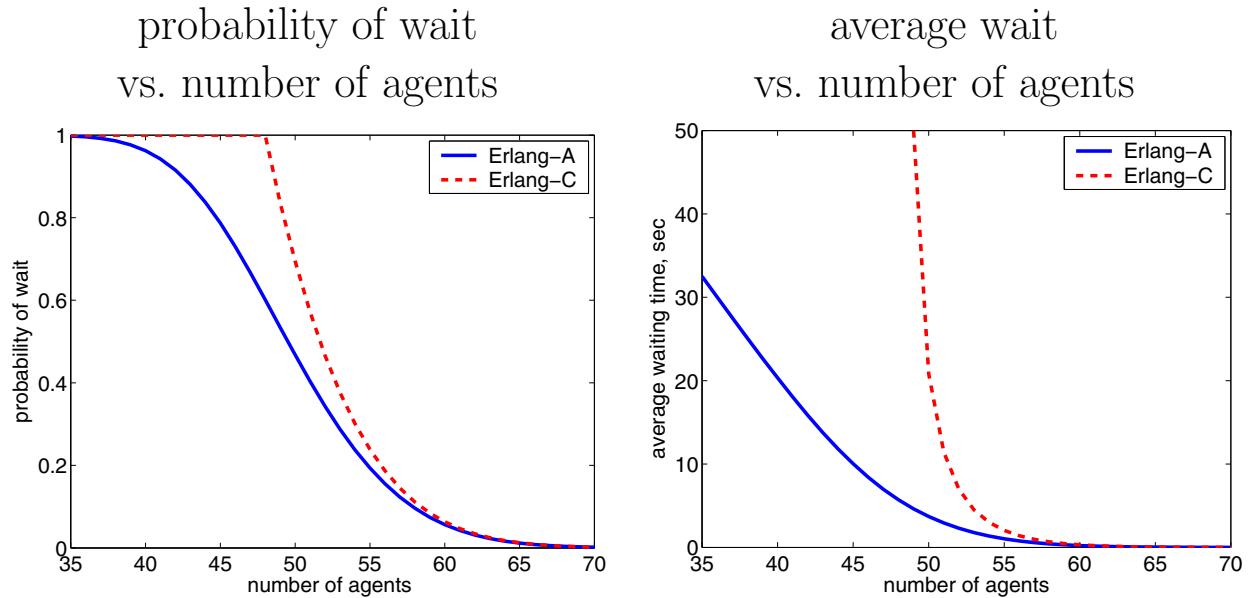
- $\lambda$  – Poisson arrival rate.
- $\mu$  – Exponential service rate.
- $n$  – number of service agents.
- $\theta$  – individual abandonment rate.



- **Patience time**  $\tau \sim \exp(\theta)$ :  
time a customer is willing to wait for service;
- **Offered wait**  $V$ :  
waiting time of a customer with infinite patience;
- If  $\tau \leq V$ , customer abandons; otherwise, gets service;
- **Actual wait**  $W_q = \min(\tau, V)$ .

## Erlang-A vs. Erlang-C

48 calls per min, 1 min average service time,  
2 min average patience



If 50 agents:

	M/M/n	M/M/n+M	M/M/n, $\lambda \downarrow 3.1\%$
Fraction abandoning	—	3.1%	—
Average waiting time	20.8 sec	3.7 sec	8.8 sec
Waiting time's 90-th percentile	58.1 sec	12.5 sec	28.2 sec
Average queue length	17	3	7
Agents' utilization	96%	93%	93%

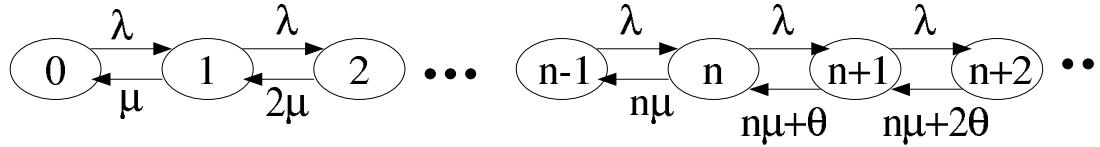
“The fittest survive” and wait less - much less.

Abandonment reduces workload when needed – at high-congestion periods.

## Erlang-A: Birth-and-Death Process

$L(t)$  – number-in-system at time  $t$  (served plus queued);  
 $L = \{L(t), t \geq 0\}$  – Markov birth-and-death process.

Transition-rate diagram



Steady-state equations:

$$\begin{cases} \lambda\pi_j = (j+1) \cdot \mu\pi_{j+1}, & 0 \leq j \leq n-1 \\ \lambda\pi_j = (n\mu + (j+1-n)\theta) \cdot \pi_{j+1}, & j \geq n. \end{cases}$$

Steady-state distribution:

$$\pi_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!} \pi_0, & 0 \leq j \leq n \\ \prod_{k=n+1}^j \left( \frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \pi_0, & j \geq n+1, \end{cases}$$

where

$$\pi_0 = \left[ \sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!} + \sum_{j=n+1}^{\infty} \prod_{k=n+1}^j \left( \frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \right]^{-1}.$$

Numerical drawback: infinite sums.

# Stability

Erlang-A is always stable!

$d_j$  – death-rate in state  $j$ ,  $0 < j < \infty$ :

$$j \cdot \min(\mu, \theta) \leq d_j \leq j \cdot \max(\mu, \theta).$$

Bounds are death rates of M/M/ $\infty$  queues with service rates  $\min(\mu, \theta)$  and  $\max(\mu, \theta)$ .

Proof of stability:

$$\begin{aligned} \pi_0^{-1} &= \sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!} + \sum_{j=n+1}^{\infty} \prod_{k=n+1}^j \left( \frac{\lambda}{n\mu + (k-n)\theta} \right) \frac{(\lambda/\mu)^n}{n!} \\ &\leq \sum_{j=0}^{\infty} \frac{(\lambda/\min(\mu, \theta))^j}{j!} = e^{-\lambda/\min(\mu, \theta)}. \end{aligned}$$

(Use that  $n\mu + (k-n)\theta \geq k \min(\mu, \theta)$ .)

## Steady-state distribution via special functions (Palm):

*Gamma function:*

$$\Gamma(x) \triangleq \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

*Incomplete Gamma function:*

$$\gamma(x, y) \triangleq \int_0^y t^{x-1} e^{-t} dt, \quad x > 0, y \geq 0.$$

$$A(x, y) \triangleq \frac{xe^y}{y^x} \cdot \gamma(x, y) = 1 + \sum_{j=1}^{\infty} \frac{y^j}{\prod_{k=1}^j (x+k)}, \quad x > 0, y \geq 0.$$

Recall  $E_{1,n}$  – *blocking probability* in M/M/n/n (Erlang-B):

$$E_{1,n} = \frac{\frac{(\lambda/\mu)^n}{n!}}{\sum_{j=0}^n \frac{(\lambda/\mu)^j}{j!}} = \frac{(\lambda/\mu)^n}{e^{\lambda/\mu}} \cdot \frac{1}{\Gamma(n+1) - \gamma(n+1, \lambda/\mu)}.$$

Can be calculated also via recursion.

Then one can show:

$$\pi_j = \begin{cases} \pi_n \cdot \frac{n!}{j! \cdot \left(\frac{\lambda}{\mu}\right)^{n-j}}, & 0 \leq j \leq n, \\ \pi_n \cdot \frac{\left(\frac{\lambda}{\theta}\right)^{j-n}}{\prod_{k=1}^{j-n} \left(\frac{n\mu}{\theta} + k\right)}, & j \geq n+1, \end{cases}$$

where

$$\pi_n = \frac{E_{1,n}}{1 + \left[ A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) - 1 \right] \cdot E_{1,n}}.$$

## Operational Performance Measures

The most popular performance measure is  $P\{W_q \leq T; \text{Sr}\}$  (or even worse  $P\{W_q \leq T | \text{Sr}\}$ ).

We recommend:

- $P\{W_q \leq T; \text{Sr}\}$  - fraction of well-served;
- $P\{\text{Ab}\}$  - fraction of poorly-served.

or a four-dimensional refinement:

- $P\{W_q \leq T; \text{Sr}\}$  - fraction of well-served;
- $P\{W_q > T; \text{Sr}\}$  - fraction of served, with potential for improvement (say, a higher priority on next visit);
- $P\{W_q > \epsilon; \text{Ab}\}$  - fraction of poorly-served;
- $P\{W_q \leq \epsilon; \text{Ab}\}$  - fraction of those whose service-level is undetermined.

### Properties of $P\{\text{Ab}\}$ :

- $P\{\text{Ab}\}$  increases monotonically in  $\theta, \lambda$ ;  
 $P\{\text{Ab}\}$  decreases monotonically in  $n, \mu$   
(Bhattacharya and Ephremides (1991))
- M/M/n+G: if  $E[\tau]$  is fixed, deterministic patience minimizes  $P\{\text{Ab}\}$  (Mandelbaum and Zeltyn (2004))

## Additional important performance measures:

- Delay probability  $P\{W_q > 0\}$ ;
- Average wait  $E[W_q]$ ;
- ASA (Average Speed of Answer) – used extensively in call centers; usually defined as  $E[W_q | \text{Sr}]$ ;
- Agents' occupancy  $\rho = \frac{\lambda \cdot (1 - P\{\text{Ab}\})}{n\mu}$ .
- Average queue-length  $E[L_q]$ .

## Operational Performance Measures: calculation via 4CallCenters

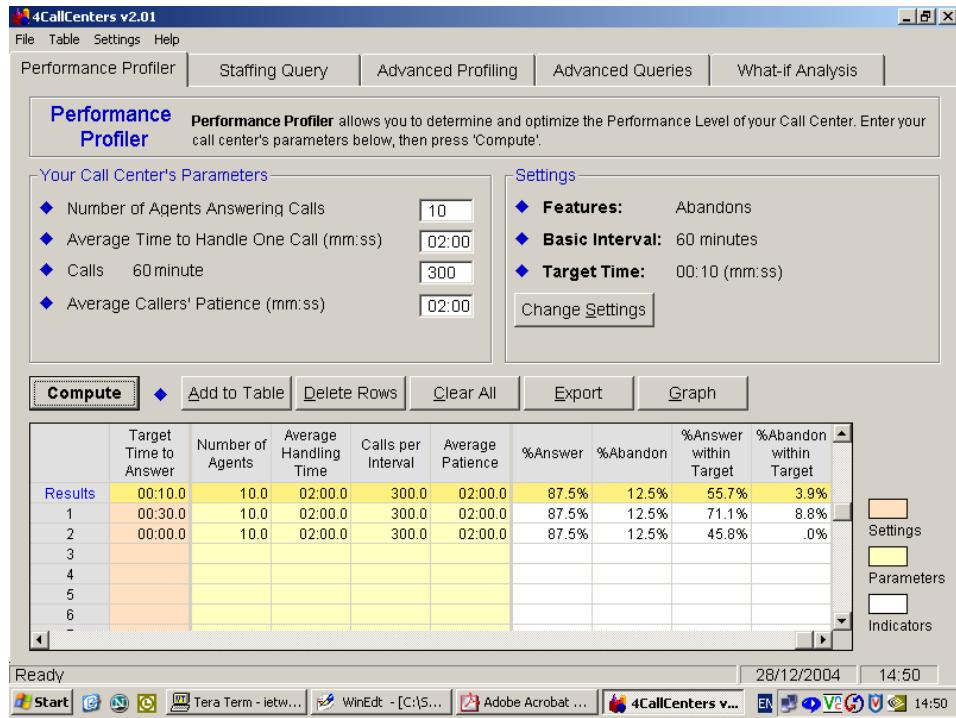
Performance measures of the form  $E[f(V, \tau)]$ .

Calculable, in numerically stable procedures.

For example,

$f(v, \tau)$	$E[f(V, \tau)]$
$1_{\{v > \tau\}}$	$P\{V > \tau\} = P\{\text{Ab}\}$
$1_{(t, \infty)}(v \wedge \tau)$	$P\{W_q > t\}$
$1_{(t, \infty)}(v \wedge \tau)1_{\{v > \tau\}}$	$P\{W_q > t; \text{Ab}\}$
$(v \wedge \tau)1_{\{v > \tau\}}$	$E\{W_q; \text{Ab}\}$
$g(v \wedge \tau)$	$E[g(W_q)]$

# Operational Performance Measures: calculation via 4CallCenters



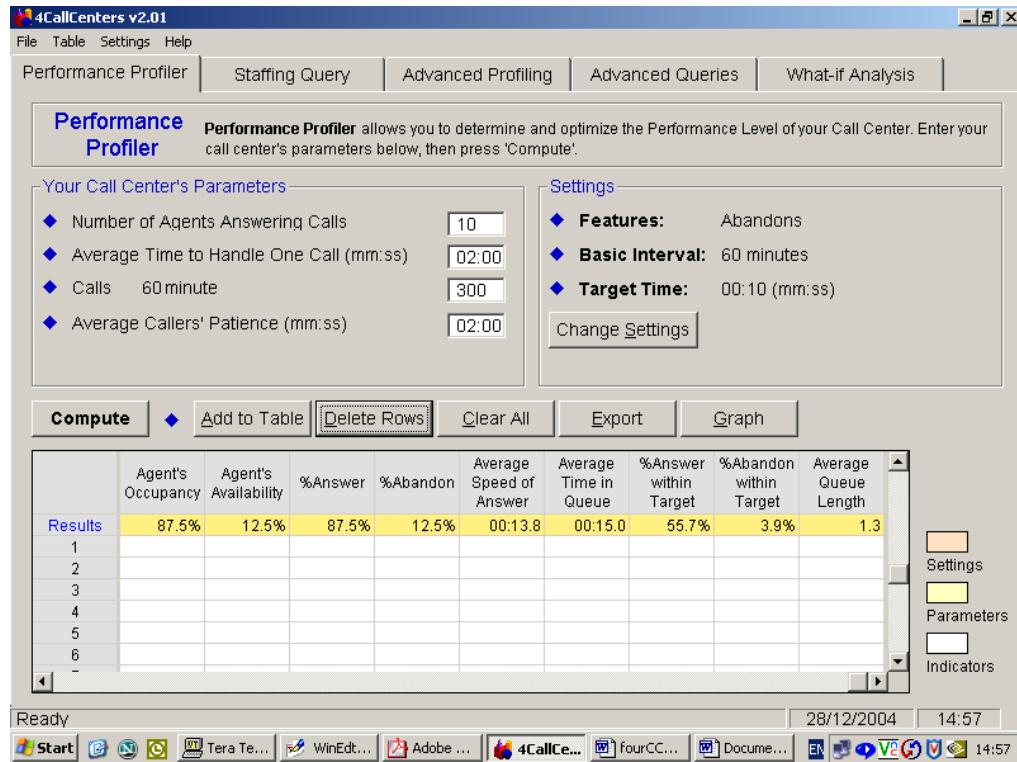
Erlang-A parameters:

$\lambda = 300$  calls per hour,  $1/\mu = 2$  min,  $n = 10$ ,  $1/\theta = 2$  min.

Target times  $T = 30$  sec,  $\epsilon = 10$  sec.

- $P\{W_q \leq T; \text{Sr}\} = 71.1\%$ ;
- $P\{W_q > T; \text{Sr}\} = 87.5\% - 71.1\% = 16.4\%$ ;
- $P\{W_q > \epsilon; \text{Ab}\} = 12.5\% - 3.9\% = 8.6\%$ ;
- $P\{W_q \leq \epsilon; \text{Ab}\} = 3.9\%$ .
- Delay probability  $P\{W_q > 0\} = 100\% - 45.8\% = 54.2\%$ .

## Additional performance measures



- Average Time in Queue =  $E[W_q] = 15$  sec;
- ASA =  $E[W_q | \text{Sr}] = 13.8$  sec;
- Agents' Occupancy  $\rho = 87.5\%$ ;
- Average Queue Length  $E[L_q] = 1.3$ .

## Operational Performance Measures: calculation via special functions

For example,

$$\begin{aligned}
 P\{W_q > 0\} &= \sum_{j=n}^{\infty} \pi_j = \frac{A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) \cdot E_{1,n}}{1 + \left(A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right) - 1\right) \cdot E_{1,n}}, \\
 P[Ab|W_q > 0] &= \frac{1}{\rho A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right)} + 1 - \frac{1}{\rho}, \\
 E[W_q|W_q > 0] &= \frac{1}{\theta} \cdot \left[ \frac{1}{\rho A\left(\frac{n\mu}{\theta}, \frac{\lambda}{\theta}\right)} + 1 - \frac{1}{\rho} \right].
 \end{aligned}$$

## Operational Performance Measures: calculation via M/M/n+G formulae

M/M/n+G – generalization of Erlang-A, patience times distributed with cdf  $G(\cdot)$ . See

<http://iew3.technion.ac.il/serveng/References/references.html>

- Mandelbaum A. and Zeltyn S. (2004) M/M/n+G queue. Summary of performance measures;
- Zeltyn S. (2004) Call centers with impatient customers: exact analysis and many-server asymptotics of the M/M/n+G queue, Ph.D. Thesis.

Explained how to adapt M/M/n+G to Erlang-A:

$$G(x) = 1 - e^{-\theta x}, \quad \theta > 0.$$

## The relation $P\{\text{Ab}\}/E[W_q]$

**Theoretical:** In Erlang-A (and other queueing models with  $\exp(\theta)$  patience):

$$P\{\text{Ab}\} = \theta \cdot E[W_q].$$

**Proof.** Balance equation:

$$\theta \cdot E[L_q] = \lambda \cdot P\{\text{Ab}\}. \quad (1)$$

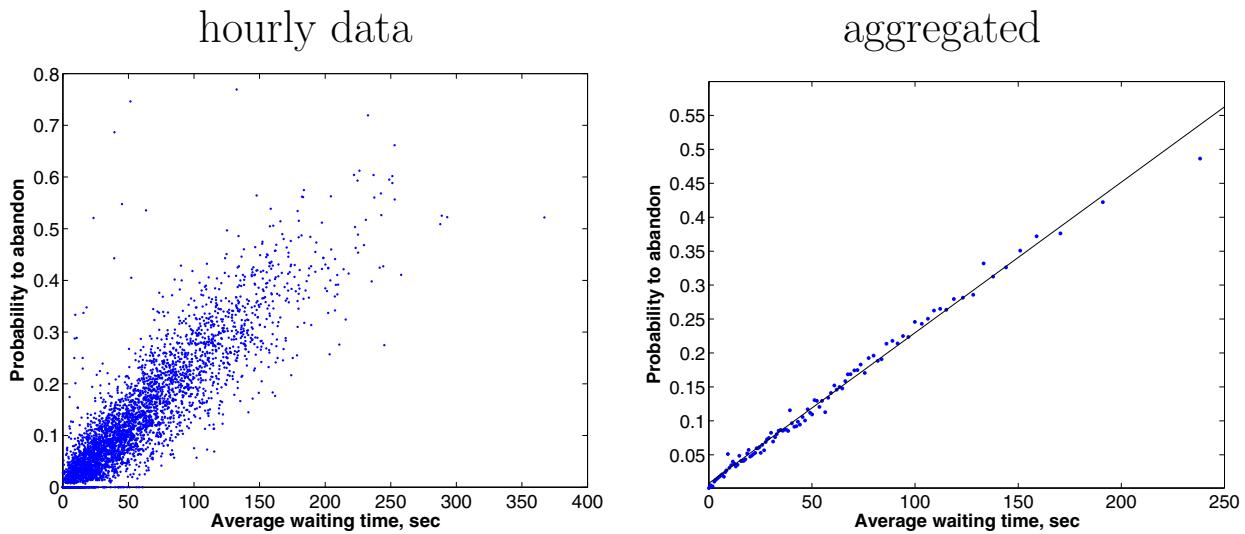
Little's formula:

$$E[L_q] = \lambda \cdot E[W_q]. \quad (2)$$

Substitute (2) into (1). ■

## Empirical relations

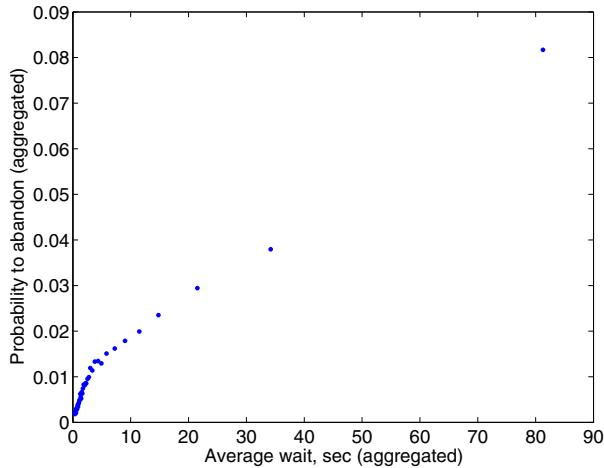
### Israeli bank: yearly data



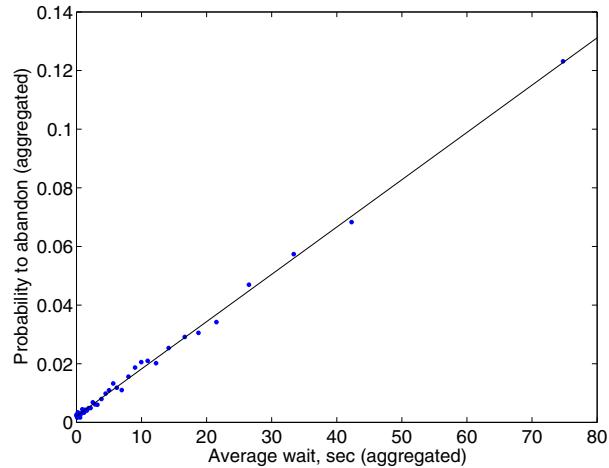
The graphs are based on 4158 hour intervals.

## U.S. bank

Retail



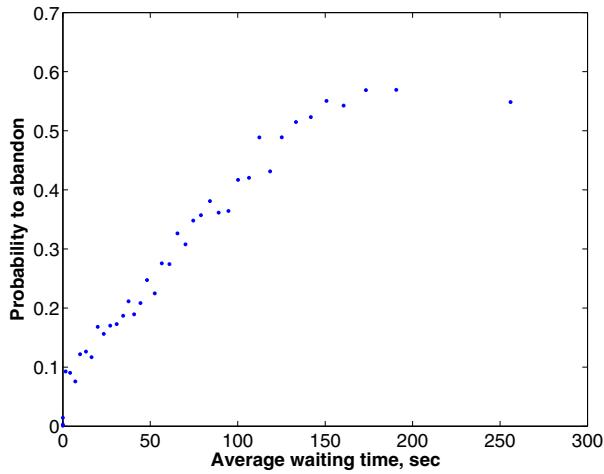
Telesales



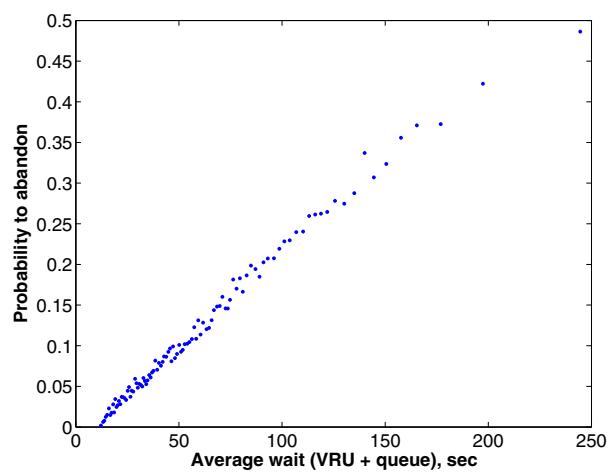
Retail – significant abandonment during first seconds of wait.

## Linear patterns with non-zero intercepts

Israeli data: new customers



VRU-time included in wait



Left-hand plot  $\approx$  exp patience with balking:

0 with probability  $p$ ,  $\exp(\theta)$  with probability  $(1 - p)$ .

Right-hand plot  $\approx$  delayed patience:  $c + \exp(\theta)$ ,  $c > 0$ .

## Erlang-A: parameter estimation and prediction

**Estimation:** inference from historical data (e.g. exp, normal) were parameters assumed fixed over time.

**Prediction:** forecast behavior of sample outside of original data set.

### Arrivals ( $\lambda$ )

- Typically Poisson, time-varying rates, constant at 15/30/60 min scale;
- Significant uncertainty concerning future rates  $\Rightarrow$  prediction;
- Predict separately *daily volumes* and *fraction* of arrivals per time interval.

### Services ( $\mu$ )

- Typically stable from day to day  $\Rightarrow$  estimation;
- Can change depending on time-of-day;
- Typically, service time  $\neq$  talk time.

#### First approach:

service time = talk time + wrap-up time (after-call work) + . . . ;

#### Second approach:

$$\text{service time} = \frac{\text{Total Working Time} - \text{Total Idle Time}}{\text{Number of Served Customers}}.$$

## Number of agents (n)

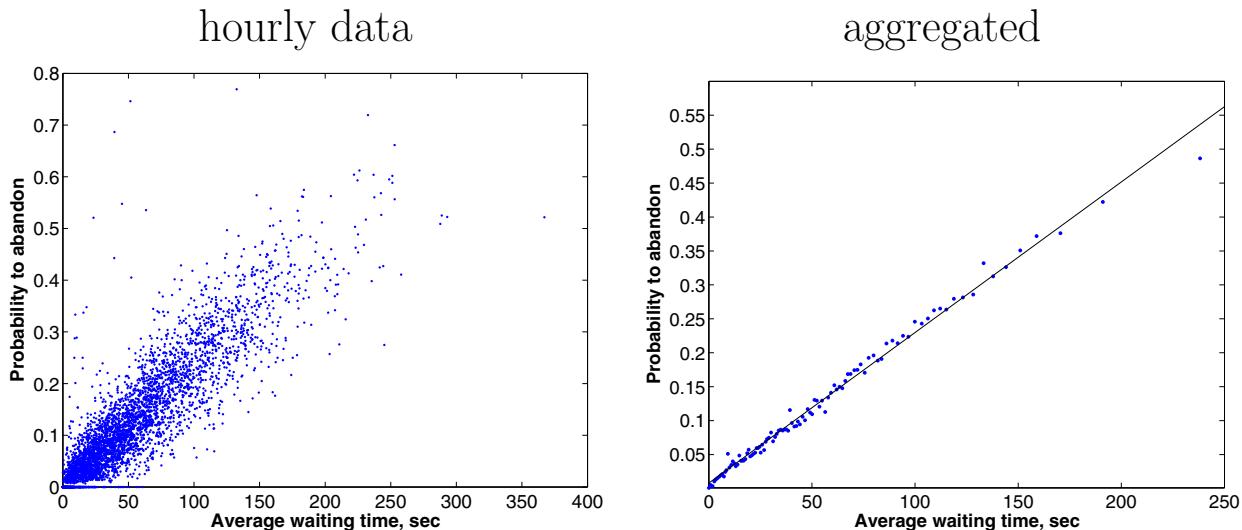
- Output of WFM software given  $\lambda, \mu, \theta$ , performance goals.  
One gets number of FTE's (Full Time Equivalent positions).
- Agents on schedule = FTE's  $\cdot$  RSF (Rostered Staff Factor) ( $RSF > 1$ ). Reasons: absenteeism, unscheduled breaks, ...
- Obtaining historical data on  $n$  can be hard.

## Patience ( $\theta$ )

Observations are **censored!** (heavily)

- Customer abandoned  $\Rightarrow$  patience  $\tau$  known;
- Customer served  $\Rightarrow$  offered wait  $V$  known  $\Rightarrow \tau > V$ .

Avoiding direct “uncensoring”: use  $P\{Ab\} = \theta \cdot E[W_q]$ .



Regression  $\Rightarrow$  average patience ( $1/\theta$ )  $\approx \frac{250}{0.56} \approx 446$  sec.

# Estimating patience distribution

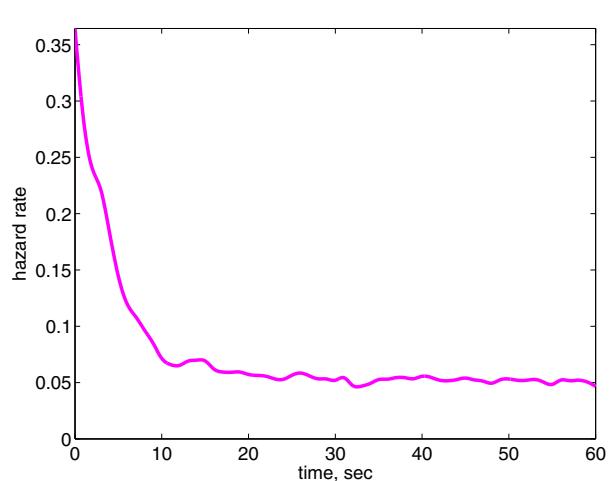
Are patience times really exponential?

To “uncensor data” use Kaplan-Meier (product-limit) estimator.

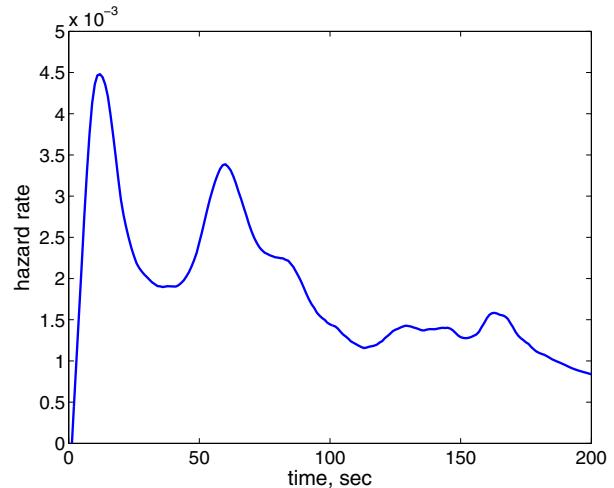
Output: estimates of survival function and hazard rate.

## Empirical hazard rates of patience times

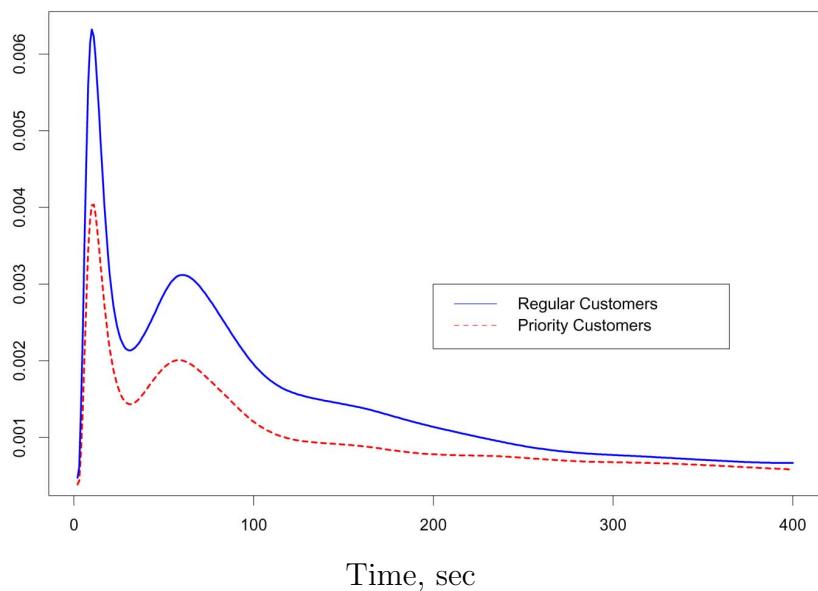
U.S. bank



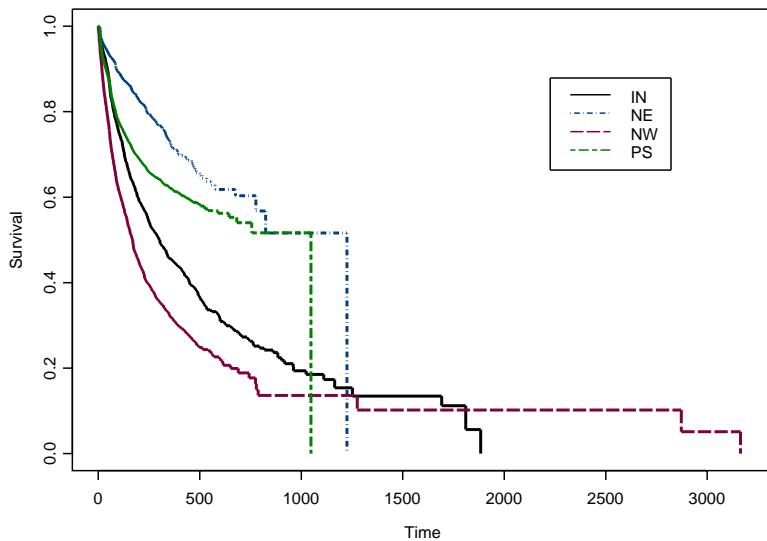
Israeli bank



## Israeli bank: regular vs. priority customers



## Israeli bank: service types



IN – Internet Assistance; NE – Stock Transactions;  
NW – New Customers; PS – Regular

### Conclusions:

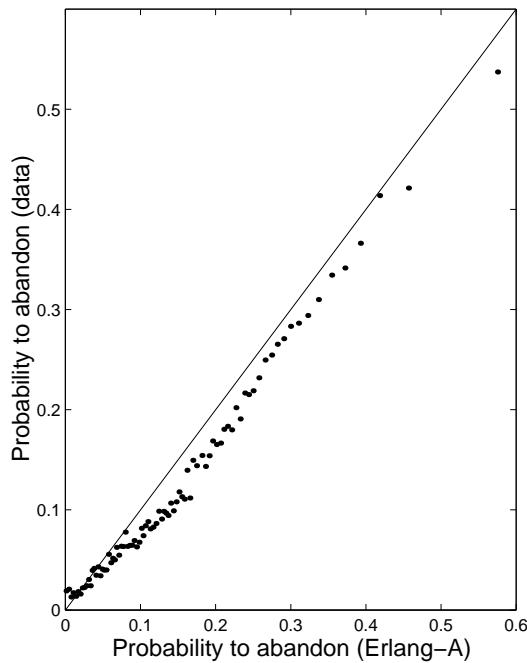
- Patience times are, in general, non-exponential;
- Most tele-customers are **very** patient;
- Kaplan-Meier is very informative concerning patience *qualitative* patterns (abandonment peaks, comparisons, . . . );
- Kaplan-Meier can be problematic concerning estimation of *quantitative* characteristics (mean, variance, median).  
 $E[\tau] = \int_0^\infty S(x)dx$ , where  $S(x)$  - survival function of patience.  
However,  $\widehat{S(x)}$  not reliable for large  $x$ .

**Question:** can we apply Erlang-A with non-exponential patience?

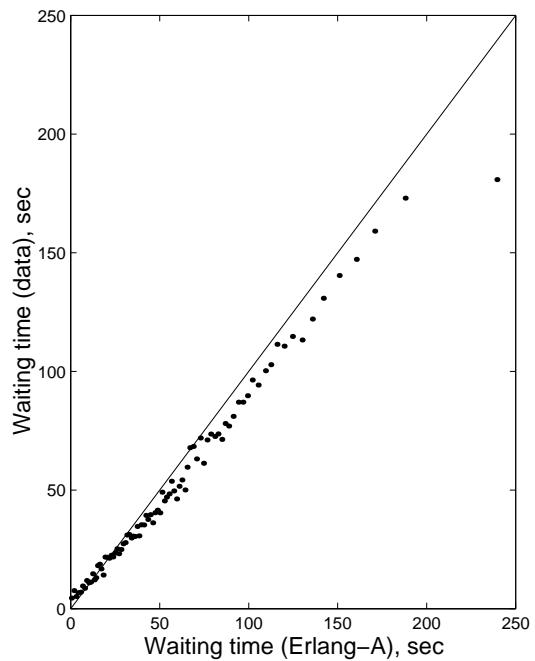
# Fitting a simple model to a complex reality

## Erlang-A Formulae vs. Data Averages (Israeli Bank)

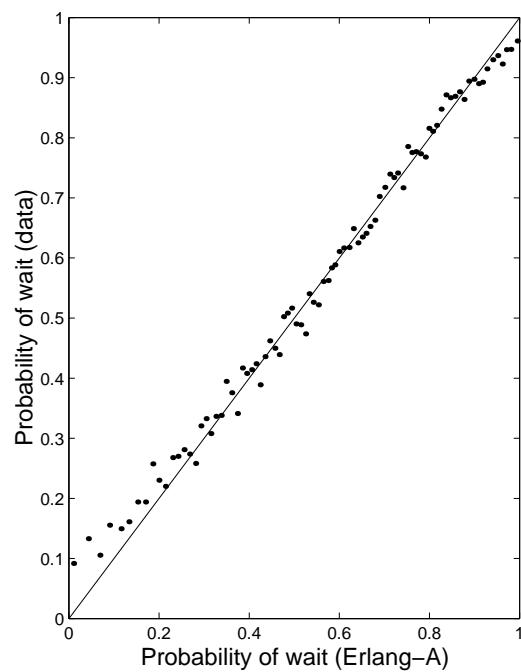
$P\{Ab\}$



$E[W_q]$



$P\{W_q > 0\}$



## Conclusions:

- Points: hourly data vs. Erlang-A output;
- Formulae with continuous  $n$  used;
- Patience estimated via  $P\{\text{Ab}\}/E[W_q]$  relation;
- Erlang-A estimates – close upper bounds.

## Fitting a simple model to a complex reality: Patience index

How to define (im)patience?

$$\begin{aligned} \text{Theoretical Patience Index} &\triangleq \frac{\text{time willing to wait}}{\text{time required to wait}} \\ &= \frac{\text{average patience}}{\text{average offered wait}}. \end{aligned} \quad (3)$$

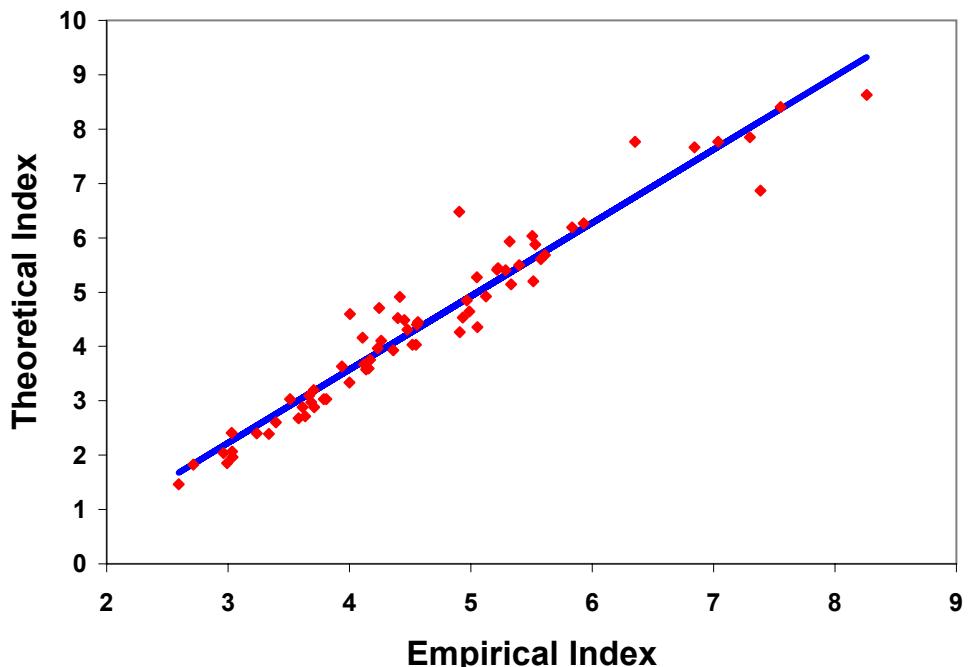
Calculation can be difficult.

$$\text{Empirical Patience Index} \triangleq \frac{\% \text{ served}}{\% \text{ abandoned}}. \quad (4)$$

Easily calculable from ACD reports.

If  $\tau$  and  $V$  exponentially distributed, (4) is MLE of (3).

### Patience index – empirical vs. theoretical



# PATIENCE INDEX

- How to Define? Measure? Manage?

<u>Statistics</u>	<u>Time Till</u>	<u>Interpretation</u>
360K served (80%)	2 min.	? must = <b>expect</b>
90K abandon (20%)	1 min.	? <b>willing</b> to wait

“Time willing to wait” of served is **censored** by their “wait”.

## “Uncensoring” (simplified)

**Willing to wait**  $1 + 2 \times \frac{360\text{K}}{90\text{K}} = 1 + 2 \times 4 = 9 \text{ min.}$

**Expect to wait**  $2 + 1 \times \frac{90\text{K}}{360\text{K}} = 2 + 1 \times \frac{1}{4} = 2.25 \text{ min.}$

$$\text{Patience Index} = \frac{\text{time willing}}{\text{time expect}} = 4 = \frac{\# \text{ served}/\text{wait} > 0}{\# \text{ abandon}/\text{wait} > 0}$$

$\uparrow$   $\uparrow$   
 definition measure

# Customer-Focused Queueing Theory

Waiting experience can be summarized by:

1. Time that a customer *expects* to wait;
2. Time that a customer is *willing* to wait ( $\tau$ , patience or need);
3. Time that a customer *must* wait ( $V$ , offered wait);
4. Time that a customer *actually* waits ( $W_q = \min(\tau, V)$ );
5. Time that a customer *perceives* waiting.

Experienced customers  $\Rightarrow 1=3$ ;

Rational customers  $\Rightarrow 4=5$ ;

Then left with  $(\tau, V, W_q)$ , as introduced before.

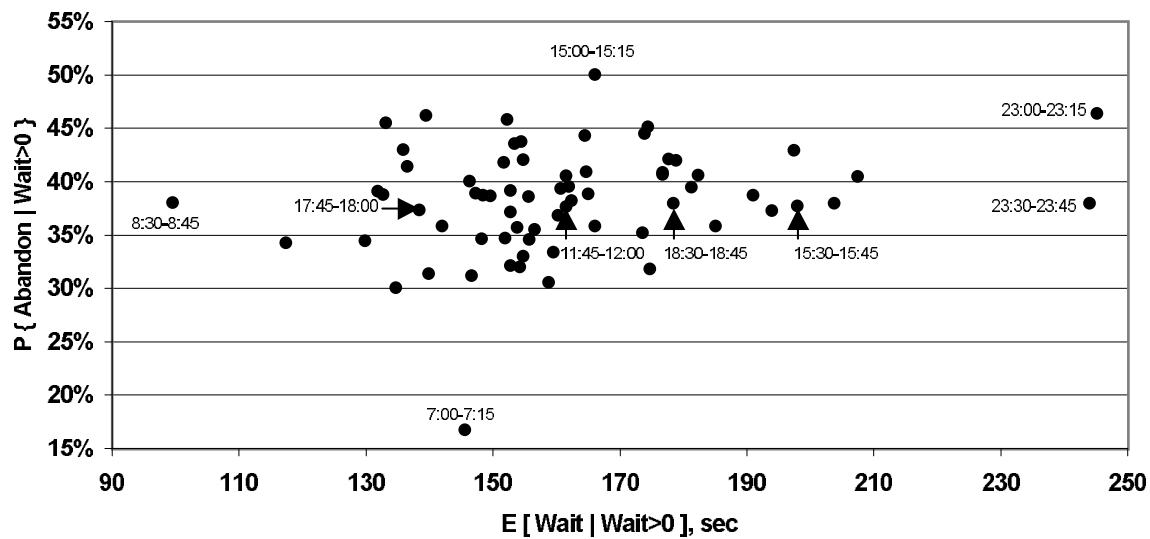
200 abandonment in Direct-Banking: perceived vs. actual waiting.

Reason to Abandon	Actual Abandon Time (sec)	Perceived Abandon Time (sec)	Perception Ratio
Fed up waiting (77%)	70	164	2.34
Not urgent (10%)	81	128	1.6
Forced to (4%)	31	35	1.1
Something came up (6%)	56	53	0.95
Expected call-back (3%)	13	25	1.9

# Adaptive behavior of impatient customers

**Question:** Do customers adapt their patience to system performance (offered wait)?

## Israeli bank: Internet-support customers



Rational abandonment from invisible queues: Mandelbaum, Shimkin, Zohar.

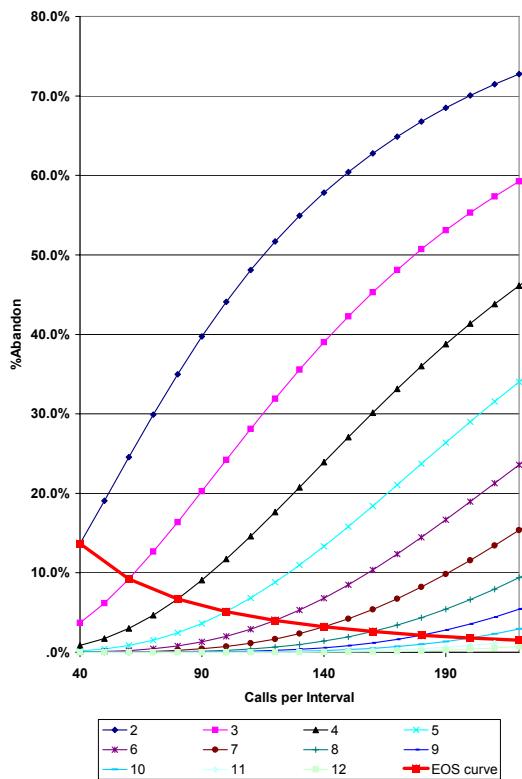
# Advanced features of 4CallCenters

## Advanced profiling

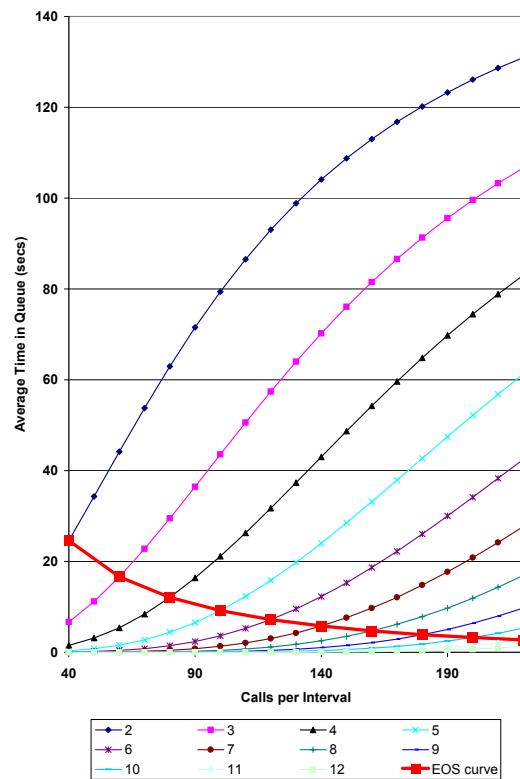
Vary input parameters of Erlang-A and display output (performance measures) in a table or graphically.

**Example:**  $1/\mu = 2$  minutes,  $1/\theta = 3$  minutes;  
 $\lambda$  varies from 40 to 230 calls per hour, in steps of 10;  
 $n$  varies from 2 to 12.

Probability to abandon



Average wait



Red curve: EOS (Economies-Of-Scale).

Why the two graphs are similar?

## Advanced staffing queries

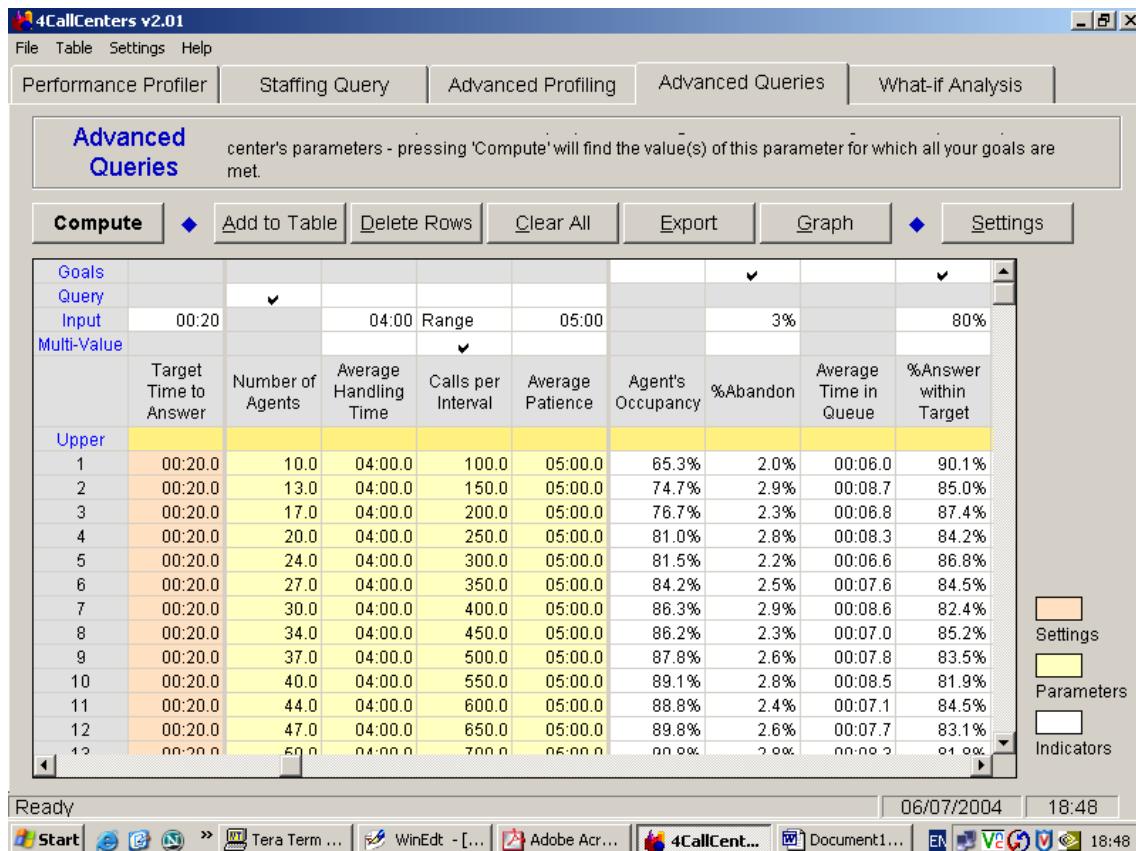
Multiple performance goals.

**Example:**  $1/\mu = 4$  minutes,  $1/\theta = 5$  minutes;  
 $\lambda$  varies from 100 to 1200, in steps of 50.

**Performance targets:**

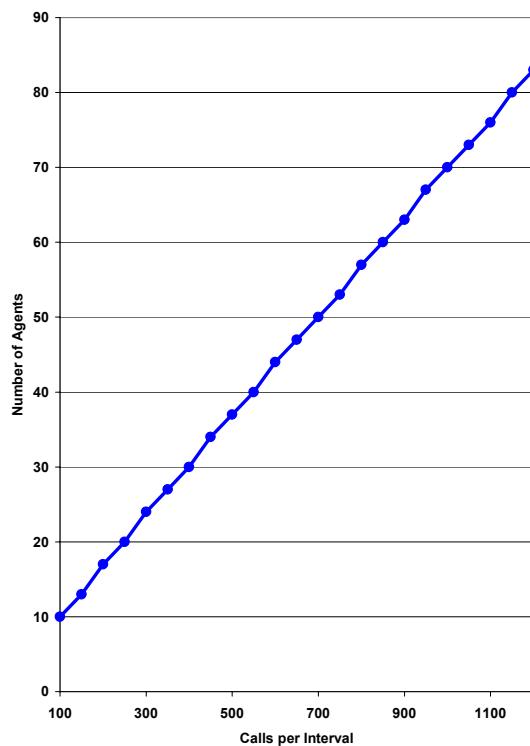
$$P\{Ab\} \leq 3\%; \quad P\{W_q < 20 \text{ sec}\}; \quad Sr \geq 0.8.$$

## 4CallCenters output

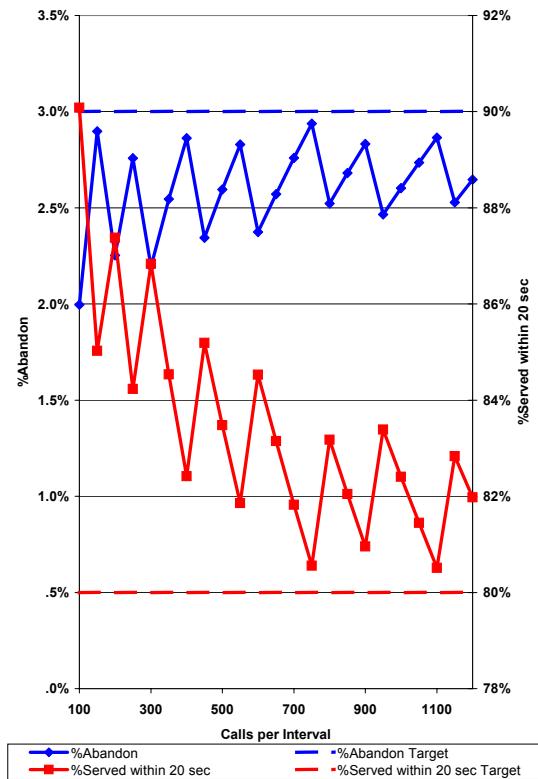


## 4CallCenters. Advanced staffing queries. Dynamics of staffing level and performance.

Recommended staffing level



Target performance measures



**EOS:** 10 agents needed for 100 calls per hour but only 83 for 1200 calls per hour.