Online Supplement for "Minimizing Mortality in a Mass Casualty Event: Fluid Networks in Support of Modeling and Staffing"

Proposition 1: The optimal solutions for Problem (P1) are the same as those for Problem (P2).

Proof: By adding the two constraints of the type $N_i(t) \leq Q_i(t) \quad \forall t \in [0,T)$ the following transformation $(Q_i(t) \wedge N_i(t)) = N_i(t)$ is made. Adding more resources than necessary (e.g. $N_i(t) > Q_i(t)$ for some t) does not affect the departure rate from a station so the objective value is not affected and the optimal solutions of (P1) and (P2) are equivalent.

Proposition 2: The formulation of (P2) is equivalent to the following formulation (P3).

Proof: We start with (P2) and perform the following steps: change the summation index so that the objective includes the terms $Q_i(t+1)$ which are substituted by the flow constraints in the objective function to yield the following:

$$\underset{N_1(\cdot),\, N_2(\cdot)}{Min} \quad \sum_{t=0}^{T-1} \{(\theta_1[(1-\theta_1)Q_1(t) + \lambda(t) - \mu_1 N_1(t)] + \theta_2[(1-\theta_2)Q_2(t) + p_{12} \cdot \mu_1 N_1(t) - \mu_2 N_2(t)]\}$$

such that for $t = 0,1,...,T-1$:

$Q_1(t+1) = (1-\theta_1)Q_1(t) + \lambda(t) - \mu_1 N_1(t)$

$Q_2(t+1) = (1-\theta_2)Q_2(t) + p_{12} \cdot \mu_1 N_1(t) - \mu_2 N_2(t)$

$N_1(t) \leq Q_1(t)$

$N_2(t) \leq Q_2(t)$

$N_1(t) + N_2(t) \leq N$

$N_1(t),\ N_2(t),\ Q_1(t),\ Q_2(t) \geq 0,$ and

$Q_1(0) = 0,\ \ Q_2(0) = 0.$

Next we continue to substitute the flow constraints which lead to sums of geometrical progression equations, place the initial conditions $Q_1(0)=0,\ \ Q_2(0)=0$ in the objective and omit constants from the objective to receive the following formulation:

$$\underset{N_1(\cdot),\,N_2(\cdot)}{Min}\ \sum_{t=1}^{T-1}\{N_1(t)\mu_1[(1-\theta_1)^{T-t}-1-\ p_{12}[(1-\theta_2)^{T-t}-1]]+\ N_2(t)\mu_2[(1-\theta_2)^{T-t}-1]\}$$

subject to

$N_1(1) = 0$

$\mu_1 N_1(1) + N_1(2) \ \le\ \lambda(1)$

$(1-\theta_1)\mu_1 N_1(1) + \mu_1 N_1(2)\ + N_1(3)\ \le\ \ (1-\theta_1)\lambda(1)\ + \lambda(2)$

$\vdots$

$(1-\theta_1)^{T-3}\mu_1 N_1(1) + (1-\theta_1)^{T-4}\mu_1 N_1(2)+\cdots +N_1(T-1)\le\ (1-\theta_1)^{T-3}\lambda(1)\ + (1-\theta_1)^{T-4}\lambda(2)+\cdots+\lambda(T-1)$

$N_2(1) = 0$

$\mu_2 N_2(1)$ -$\mathrm{p}_{12}\mu_1 N_1(1)+ N_2(2)\le 0$

$(1-\theta_2)\mu_2 N_2(1)$ -$(1-\theta_2)\mathrm{p}_{12}\mu_1 N_1(1)+ \mu_2 N_2(2)$ - $\mathrm{p}_{12}\mu_1 N_1(2)+ N_2(3)\le 0$

$\vdots$

$(1-\theta_2)^{T-3}\mu_2 N_2(1) + (1-\theta_2)^{T-3}\mathrm{p}_{12}\mu_1 N_1(1)+(1-\theta_2)^{T-4}\mu_2 N_2(2)$ - $(1-\theta_2)^{T-4}\mathrm{p}_{12}\mu_1 N_1(2)+\cdots$

$\cdots+\mu_2 N_2(T-2)$ - $\mathrm{p}_{12}\mu_1 N_1(T-2)$ $+N_2(T\text{-}1)\le 0$

$N_1(t)+N_2(t)\le N \quad \forall t\in\{0,T-1\}$

$N_1(t),\ N_2(t)\ \ge 0 \quad \forall t\in\{0,T-1\}$,

which is identical to (P3).

Proposition 3: An optimal policy for the greedy problem is to allocate to Station i^* all needed[1] surgeons from the *N* available, where $i^* = 1$ if $\mu_1(\theta_1 - p_{12} \cdot \theta_2) \geq \mu_2 \cdot \theta_2$ and $i^* = 2$ otherwise; if there are still available surgeons left then allocate them to the other station.

Proof: A greedy solution solves an optimization problem for each time interval *t*, determining $N_1(t),\ N_2(t)$ to minimize the mortality in the next interval (*t+1*). The optimization problem is:

$$\underset{N_1(\cdot),\ N_2(\cdot)}{Min} \quad \theta_1 Q_1(t+1) + \theta_2 Q_2(t+1) \qquad \forall t \in \{0, T-1\}$$

subject to

$$Q_1(t+1) = Q_1(t) + \lambda(t) - \mu_1 N_1(t) - \theta_1 \cdot Q_1(t)$$
$$Q_2(t+1) = Q_2(t) + p_{12} \cdot \mu_1 N_1(t) - \mu_2 N_2(t) - \theta_2 \cdot Q_2(t)$$
$$N_1(t) \leq Q_1(t)$$
$$N_2(t) \leq Q_2(t)$$
$$N_1(t) + N_2(t) \leq N$$
$$N_1(t),\ N_2(t),\ Q_1(t),\ Q_2(t) \geq 0$$
$$Q_1(0) = 0, \quad Q_2(0) = 0.$$

Substituting the flow constraints in the objective function yields the following:

$$\underset{N_1(\cdot),\ N_2(\cdot)}{Min} \quad \theta_1[(1-\theta_1)Q_1(t) + \lambda(t) - \mu_1 N_1(t)] + \theta_2[(1-\theta_2)Q_2(t) + p_{12} \cdot \mu_1 N_1(t) - \mu_2 N_2(t)] \qquad \forall t \in \{0, T-1\}$$

subject to

$$N_1(t) \leq Q_1(t)$$
$$N_2(t) \leq Q_2(t)$$
$$N_1(t) + N_2(t) \leq N$$
$$N_1(t),\ N_2(t),\ Q_1(t), Q_2(t) \geq 0$$
$$Q_1(0) = 0, \quad Q_2(0) = 0.$$

After omitting constants (e.g., at any time *t*, $Q_1(t), Q_2(t)$ and λ(t) are known) and formulating the problem as a maximization problem we get the following:

[1] By "needed" we mean that at most $Q_1(t)$ ($Q_2(t)$) are needed at station 1 (2).

$$\underset{N_1(\cdot),\, N_2(\cdot)}{Max} \quad N_1(t)\cdot\mu_1[\,\theta_1 - p_{12}\theta_2] + N_2(t)\cdot\mu_2\cdot\theta_2 \qquad \forall t \in \{0, T-1\}$$

subject to

$N_1(t) \le Q_1(t)$

$N_2(t) \le Q_2(t)$

$N_1(t) + N_2(t) \le N$

$N_1(t),\ N_2(t),\ Q_1(t),\ Q_2(t) \ge 0$

$Q_1(0) = 0, \quad Q_2(0) = 0.$

Note that this formulation is actually the continuous Knapsack problem (Kellerer et al., 2004, pp.17–20) so the optimal solution is to allocate all needed resources to Station 1 if $\mu_1(\theta_1 - p_{12}\cdot\theta_2) \ge \mu_2\cdot\theta_2$ ("needed" is enforced by the constraint $N_1(t) \le Q_1(t)$) or else prioritize Station 2; if there are still available surgeons left then allocate them to the other station.

Proposition 4: Assume that $\theta = \theta_1 = \theta_2$. Then an optimal solution of Problem (P3) is given by any sequence of greedy solutions for Problem (P4).

Proof: The proof, by induction, involves two steps: In the first we show that for any two minutes $(n = 2)$ and any initial conditions the greedy solution is optimal. Then in the second, we assume that the greedy solution is optimal for any n when the mortality rates are equal, and prove it for $n+1$.

Step 1:

The problem for the first minute is:

$$\underset{N_1(0),\, N_2(0)}{Max} \quad \mu_1\cdot(1 - p_{12})\cdot N_1(0) + \mu_2 N_2(0)$$

subject to

$N_1(0) \le Q_1(0)$

$N_2(0) \le Q_2(0)$

$N_1(0) + N_2(0) \le N$

$N_1(0),\ N_2(0) \ge 0$

and its optimal solution is:

$$\left[N_1(0), N_2(0)\right] = \begin{cases} N_1(0) = \min(Q_1(0), N), \quad N_2(0) = \min(Q_2(0), N - N_1(0)) & \text{when } (1-p_{12})\mu_1 > \mu_2 \\ N_1(0) = \min(Q_1(0), N - N_2(0)), \quad N_2(0) = \min(Q_2(0), N) & \text{when } (1-p_{12})\mu_1 < \mu_2 \end{cases}.$$

In a similar way we find the optimal solution for the second minute to be:

$$\left[N_1(1), N_2(1)\right] = \begin{cases} N_1(1) = \min(Q_1(1), N), \quad N_2(1) = \min(Q_2(1), N - N_1(1)) \text{ when } (1-p_{12})\mu_1 > \mu_2 \\ N_1(1) = \min(Q_1(1), N - N_2(1)), \quad N_2(1) = \min(Q_2(1), N) \text{ when } (1-p_{12})\mu_1 < \mu_2 \end{cases}$$

We explicitly write the problem for the two-minute time horizon as:

$$\underset{\substack{N_1(0),\, N_2(0) \\ N_1(1),\, N_2(1)}}{Max} \quad [1-p_{12}]\mu_1(2\text{-}\theta)N_1(0) + \mu_2(2\text{-}\theta)N_2(0) + [1-p_{12}]\mu_1 N_1(1) + \mu_2 N_2(1) \qquad \text{(PA)}$$

subject to

$N_1(0) + N_2(0) \le N$

$N_1(1) + N_2(1) \le N$

$N_1(0) \le Q_1(0),\ N_2(0) \le Q_2(0)$

$N_1(1) \le Q_1(1),\ N_2(1) \le Q_2(1)$

$N_1(0),\ N_2(0) \ge 0$

$N_1(1),\ N_2(1) \ge 0$

Our proof stands any initial conditions so we consider these 8 different combinations:

If $(1-p_{12})\mu_1 \ge \mu_2$:

1. $N \le Q_1(0), \quad N \le Q_1(1),$
2. $N > Q_1(0), \quad N > Q_1(1),$
3. $N \le Q_1(0), \quad N > Q_1(1),$
4. $N > Q_1(0), \quad N \le Q_1(1),$

$\forall Q_2(0), Q_2(1)$

If $(1-p_{12})\mu_1 < \mu_2$:

5. $N \le Q_2(0), \quad N \le Q_2(1),$
6. $N > Q_2(0), \quad N > Q_2(1),$
7. $N \le Q_2(0), \quad N > Q_2(1),$
8. $N > Q_2(0), \quad N \le Q_2(1),$

$\forall Q_1(0), Q_1(1)$

We show here the proof for the first combination and all the others were solved analogously:

The solution for solving each minute separately is: $N_1(0) = N,\ N_2(0) = 0\,; N_1(1) = N, N_2(1) = 0$. The objective function is: $O = N\mu_1\ (1\text{-}p_{12})(3\text{-}\theta)$.

Let us assume that it is optimal not to assign all of the resources to the first station in the first minute, $N_1(0) < N,\ N_2(0) = \min(N - N_1(0), Q_2(0))$.

Define: $N - N_1(0) = X > 0$ so $N_1(0) = N - X$.

Substituting the above in the objective function of Problem (PA) we get:

$$O' = N_1(0)(2-\theta)\mu_1(1-p_{12}) + N_2(0)(2-\theta)\mu_2 + N\mu_1(1-p_{12}) \le$$
$$N_1(0)(2-\theta)\mu_1(1-p_{12}) + (N - N_1(0))(2-\theta)\mu_2 + N\mu_1(1-p_{12}) =$$
$$N(3-\theta)\mu_1(1-p_{12}) - X(2-\theta)[\mu_1(1-p_{12}) - \mu_2] \le N(3-\theta)\mu_1(1-p_{12}) = O.$$

Since we wish to maximize the objective function, we get a contradiction to our assumption.

Let us focus on the second minute and assume that it is optimal not to assign all the resources to the first station in the second minute, $N_1(1) < N$, $N_2(1) = \min(N - N_1(1), Q_2(1))$ and define:

$N - N_1(1) = X > 0$ so $N_1(1) = N - X$.

As before, we place the above in the objective of Problem (PA) and get:

$$O' = N(2-\theta)\ \mu_1(1-p_{12}) + N_1(1)\mu_1(1-p_{12}) + N_2(1)\mu_2 \le$$
$$N(2-\theta)\ \mu_1(1-p_{12}) + N_1(1)\mu_1(1-p_{12}) + N\mu_2 - N_1(1)\mu_2 =$$
$$N(2-\theta)\ \mu_1(1-p_{12}) + N\mu_1(1-p_{12}) - N\mu_2 + N\mu_2 - X[\mu_1(1-p_{12}) - \mu_2] =$$
$$N\mu_1(1-p_{12})(3-\theta) - X[\mu_1(1-p_{12}) - \mu_2] \le N\mu_1(1-p_{12})(3-\theta) = O.$$

Again there is a contradiction of the assumption that it is best not to assign all resources to Station 1.

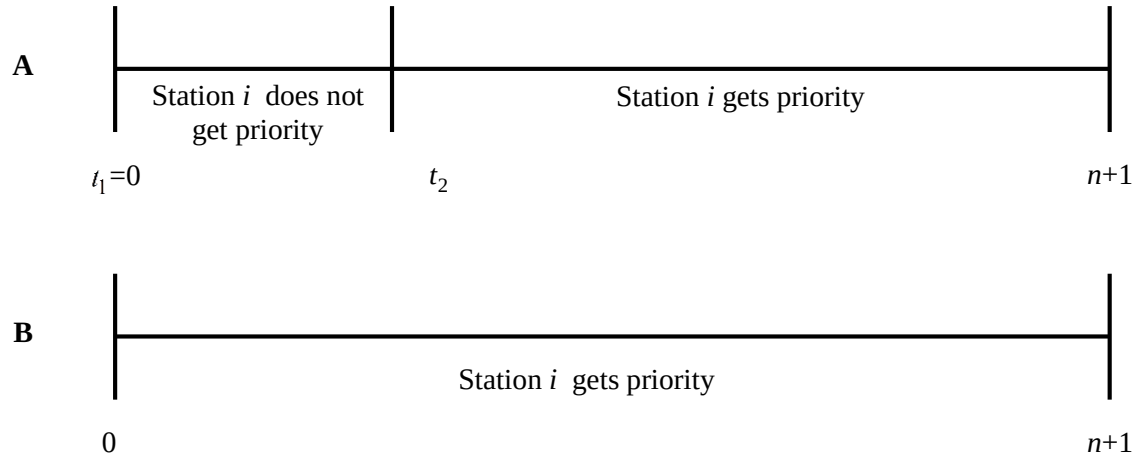
Step 2:

The induction assumption is that for n minutes (or less) the greedy solution is the optimal one and we wish to prove it for $n+1$ minutes. Let us assume, by contradiction, that in the optimal solution of $n+1$ minutes there exists a time interval: $[t_1, t_2]$, where $t_2 - t_1 \ge 1$, and in which the priority is not given to station *i*. There are four possible options regarding the location of the time interval $[t_1, t_2]$:

1. At the beginning of the MCE event $(t_1 = 0, t_2 < n+1)$.
2. At the middle of the event $(t_1 > 0, t_2 < n+1)$.
3. At the end of the event $(t_1 > 0, t_2 = n+1)$.
4. During the entire event $(t_1 = 0, t_2 = n+1)$.

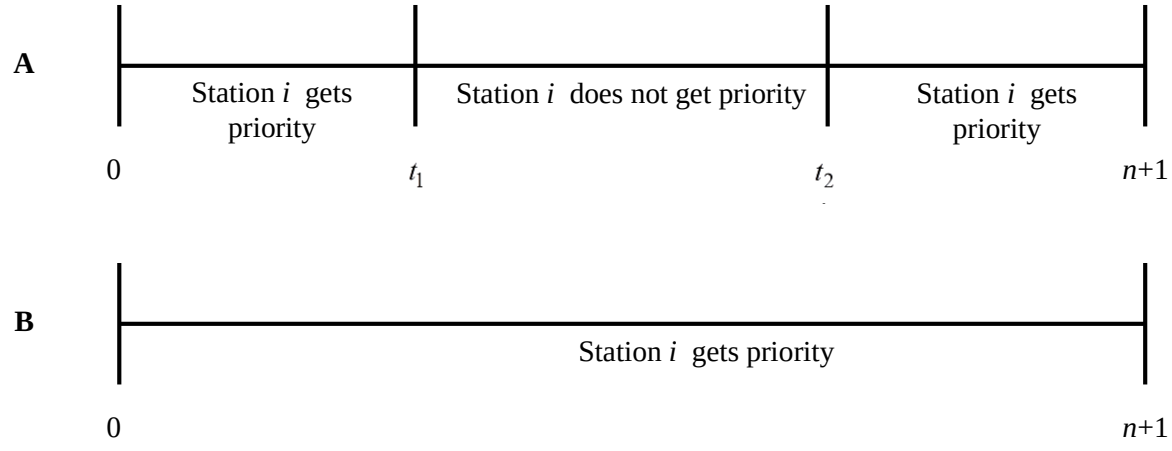
In the following figures, Option A illustrates a switch in priorities and Option B illustrates the induction's assumption that we prove to be the optimal policy.

Option 1: $t_1 = 0, t_2 < n+1$



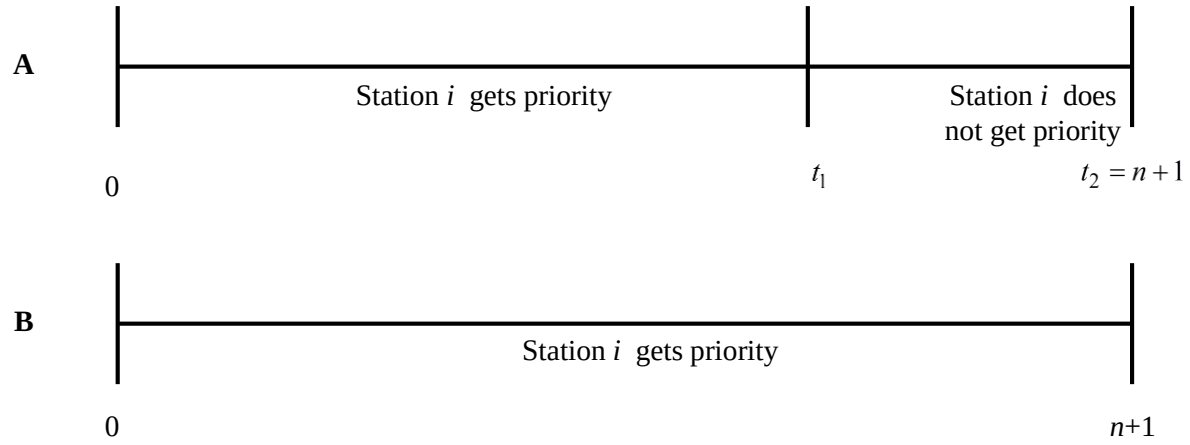
We divide the n+1 minutes into two intervals [0, t] and [t, n+1] where $t_1 < t < t_2$ and both are no longer than n minutes. In the first interval, according to the induction assumption, Option B is preferable since it gives priority to Station i. In the second interval the induction assumption also holds and therefore option B is preferable. If Option B is preferable in both intervals, it is also preferable for the entire interval.

Option 2: $t_1 > 0, t_2 < n+1$



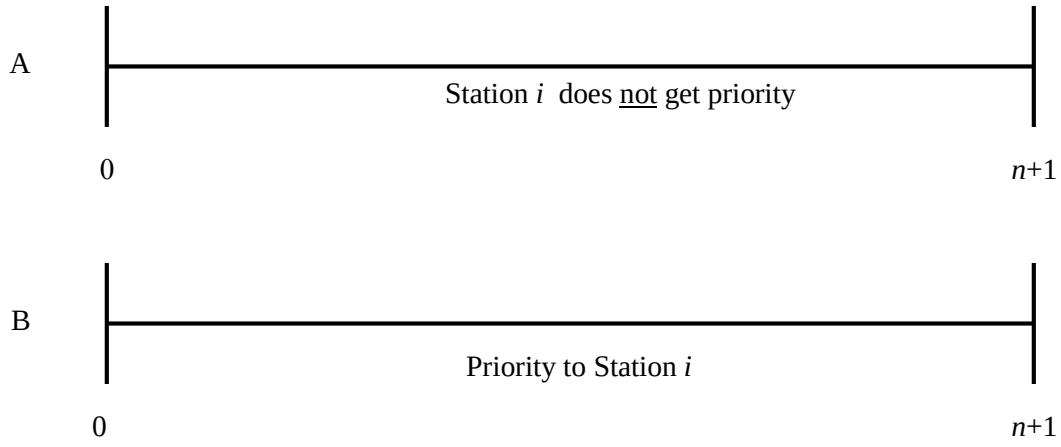
We divide the n+1 minutes into two intervals [0, t_1] and [t_1,n+1] both no longer than n minutes. In the first and last interval, the two options are identical. In the second interval, according to the induction assumption, Option B is preferable. It follows immediately that Option B is optimal for the entire interval.

Option 3: $t_1 > 0, t_2 = n+1$



We divide the *n*+1 minutes into two intervals $[0, t_1]$ and $[t_1, n+1]$ both no longer than *n* minutes. In the first interval, the two options are the same. In the second interval, according to the induction assumptions, Option B is preferable and thus, it is also preferable for the entire interval.

Option 4: $t_1 = 0, t_2 = n+1$



We divide the *n*+1 minutes into two intervals [0,*t*] and [*t*, *n*+1] both no longer than *n* minutes. In both intervals, according to the induction assumption, Option B is preferable since it gives priority to Station *i*; thus it is optimal.

Proposition 5: If Station *i* gets priority when $\theta_i = \theta_j$, then it will get priority when $\theta_i > \theta_j$ (e.g., Cases 4,5,7 and 9).

Proof: By induction. The proof involves two steps. In the first we prove the proposition for any two minutes and for any initial conditions. Then the second step expands the proof beyond two minutes.

Step 1:

The problem for the first minute is:

$$\underset{N_1(0),\, N_2(0)}{Max} \quad \mu_1[\theta_1 - \theta_2 p_{12}]N_1(0) + \theta_2\mu_2 N_2(0)$$

subject to

$$N_1(0) \le Q_1(0)$$
$$N_2(0) \le Q_2(0)$$
$$N_1(0) + N_2(0) \le N$$
$$N_1(0),\ \ N_2(0) \ge 0$$

and its optimal solution is:

$$\left[N_1(0), N_2(0)\right] = \begin{cases} N_1(0) = \min(Q_1(0), N), \quad N_2(0) = \min(Q_2(0), N - N_1(0)) \ \text{ when } (\theta_1 - \theta_2 p_{12})\mu_1 > \theta_2\mu_2 \\ N_1(0) = \min(Q_1(0), N - N_2(0)), \quad N_2(0) = \min(Q_2(0), N) \text{ when } (\theta_1 - \theta_2 p_{12})\mu_1 < \theta_2\mu_2 \end{cases}$$

Similarly, the optimal solution for the second minute is:

$$\left[N_1(1), N_2(1)\right] = \begin{cases} N_1(1) = \min(Q_1(1), N), \quad N_2(1) = \min(Q_2(1), N - N_1(1)) \ \text{ when } (\theta_1 - \theta_2 p_{12})\mu_1 > \theta_2\mu_2 \\ N_1(1) = \min(Q_1(1), N - N_2(1)), \quad N_2(1) = \min(Q_2(1), N) \text{ when } (\theta_1 - \theta_2 p_{12})\mu_1 < \theta_2\mu_2 \end{cases}.$$

Formulating the problem for the two-minute time horizon gives:

$$\underset{\substack{N_1(0),\, N_2(0) \\ N_1(1),\, N_2(1)}}{Max} \quad [\theta_1(2-\theta_1) - \theta_2 p_{12}(2\text{-}\theta_2)]\mu_1 N_1(0) + \theta_2\mu_2(2\text{-}\theta_2)N_2(0) + [\theta_1 - \theta_2 p_{12}]\mu_1 N_1(1) + \theta_2\mu_2 N_2(1)$$

subject to

$$N_1(0) + N_2(0) \le N$$
$$N_1(1) + N_2(1) \le N$$
$$N_1(0) \le Q_1(0),\ N_2(0) \le Q_2(0)$$
$$N_1(1) \le Q_1(1),\ N_2(1) \le Q_2(1)$$
$$N_1(0),\ \ N_2(0) \ge 0$$
$$N_1(1),\ \ N_2(1) \ge 0.$$

We check these 8 possible combinations of the initial conditions:

$If \quad (1-p_{12})\mu_1 \geq \mu_2$:

1. $N \leq Q_1(0), \quad N \leq Q_1(1),$
2. $N > Q_1(0), \quad N > Q_1(1),$
3. $N \leq Q_1(0), \quad N > Q_1(1),$
4. $N > Q_1(0), \quad N \leq Q_1(1),$

$\forall Q_2(0), Q_2(1)$

$If \quad (1-p_{12})\mu_1 < \mu_2$:

5. $N \leq Q_2(0), \quad N \leq Q_2(1),$
6. $N > Q_2(0), \quad N > Q_2(1),$
7. $N \leq Q_2(0), \quad N > Q_2(1),$
8. $N > Q_2(0), \quad N \leq Q_2(1),$

$\forall Q_1(0), Q_1(1)$

The proof is very long thus it is shown only for the first combination. All the others were solved analogously:

The solution for solving each minute separately is: $N_1(0) = N,\ N_2(0) = 0\,; N_1(1) = N, N_2(1) = 0$. The objective function is: $O = N\mu_1\,[\theta_1(2\text{-}\theta_1)\text{-}\ \theta_2 \mathrm{p}_{12}(2\text{-}\theta_2)] + N\mu_1\ (\theta_1 - \mathrm{p}_{12}\theta_2)$.

Let us assume that it is optimal not to assign all of the resources to the first station at the first minute

$N_1(0) < N,\ N_2(0) = \min(N - N_1(0), Q_2(0))$.

Define: $N - N_1(0) = X > 0$ so $N_1(0) = N - X$.

Substituting the above in the objective function of the problem formulation for two minutes results in:

$$
\begin{aligned}
O' &= N_1(0)\mu_1[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] + N_2(0)(2-\theta_2)\theta_2\mu_2 + N\mu_1(\theta_1 - \theta_2 p_{12}) \ \leq \\
&= N_1(0)\mu_1[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] + N(2-\theta_2)\theta_2\mu_2 - N_1(0)(2-\theta_2)\theta_2\mu_2 + \\
&\quad + N\mu_1(\theta_1 - \theta_2 p_{12}) = \\
&= N_1(0)\mu_1\{[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] - (2-\theta_2)\theta_2\mu_2\} + N(2-\theta_2)\theta_2\mu_2 + \\
&\quad + N\mu_1(\theta_1 - \theta_2 p_{12}) = \\
&= N\{\mu_1[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] - (2-\theta_2)\theta_2\mu_2\} + N(2-\theta_2)\theta_2\mu_2 + \\
&\quad + N\mu_1(\theta_1 - \theta_2 p_{12}) - X\{[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] - (2-\theta_2)\theta_2\mu_2\} = \\
&= N\mu_1[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] + N\mu_1(\theta_1 - \theta_2 p_{12}) - \\
&\quad - X\{[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] - (2-\theta_2)\theta_2\mu_2\} = \\
&= O - X\{\mu_1[\theta_1(2\text{-}\ \theta_1) - \theta_2 \mathrm{p}_{12}(2\text{-}\ \theta_2)] - (2-\theta_2)\theta_2\mu_2\} = \\
&= O - X[\theta_1(2\text{-}\ \theta_1)\mu_1 - \theta_2(2\text{-}\ \theta_2)(\mu_2 + \mu_1 p_{12})] \ \leq \ O.
\end{aligned}
$$

The coefficient of X is positive due to:

1. $\mu_1 - (\mu_2 + \mu_1 p_{12}) \geq 0$ since $\mu_1(1 - p_{12}) \geq \mu_2$

2. $\theta_1(2- \theta_1) - \theta_2(2- \theta_2) \geq 0$ when $\theta_1 > \theta_2$ *and* $(\theta_1 + \theta_2) \leq 2$ since
   $2\theta_1 - \theta_1^2 \geq 2\theta_2 - \theta_2^2$
   $2(\theta_1 - \theta_2) \geq \theta_1^2 - \theta_2^2 = (\theta_1 - \theta_2)(\theta_1 + \theta_2)$
   when $\theta_1 > \theta_2$
   $2 \geq (\theta_1 + \theta_2)$.

We note that θ_1 and θ_2 are much smaller than 1 according to the model's assumptions.

Since we wish to maximize the objective function we get a contradiction to our assumption and so acting by the greedy algorithm is optimal. The proof is by the same procedure that is detailed in Proposition 4.

Analysis of Cases 6 and 8:

Since it is not possible to fully characterize the policy for Cases 6 and 8 (e.g., at what point in time should the priority be switched), and decisions have to be made on a case-by-case basis, we provide in the sequel insights into the conditions in which we expect a greedy allocation policy, that prioritizes a single station, to perform well. Obviously, the simpler static priority setting is attractive as its solution value (e.g., the number of dead) is close to the solution value of Problem (P3). Although we cannot develop bounds, we are able to find conditions under which the static priority achieves the best and worst results. So for a given scenario one can estimate how well the static priority rule will perform. These results are presented as Corollary 1.

Corollary 1: The largest difference between the greedy and optimal solutions of Cases 6 and 8 occur for the following $\frac{\theta_2}{\theta_1}$ values:

1) Case 6:
$$\begin{cases} \text{6a: if } \mu_1(\theta_1 - \theta_2 p_{12}) > \mu_2\theta_2 \text{ then } \dfrac{\theta_2}{\theta_1} = \dfrac{\mu_1}{\mu_2 + \mu_1 p_{12}} - \varepsilon \\ \text{6b: if } \mu_1(\theta_1 - \theta_2 p_{12}) < \mu_2\theta_2 \text{ then } \dfrac{\theta_2}{\theta_1} = \dfrac{\mu_1}{\mu_2 + \mu_1 p_{12}} + \varepsilon \end{cases}$$

2) Case 8:
$$\begin{cases} \text{8a: if } \mu_1(\theta_1 - \theta_2 p_{12}) < \mu_2\theta_2 \text{ then } \dfrac{\theta_2}{\theta_1} = \dfrac{\mu_1}{\mu_2 + \mu_1 p_{12}} + \varepsilon \\ \text{8b: if } \mu_1(\theta_1 - \theta_2 p_{12}) > \mu_2\theta_2 \text{ then } \dfrac{\theta_2}{\theta_1} = \dfrac{\mu_1}{\mu_2 + \mu_1 p_{12}} - \varepsilon \end{cases}$$

where ε is positive and small enough.

Proof:

The optimal solution for Cases 6 and 8 differs from the greedy solution. Next we identify the cases when the greedy solution performs the worst; *t* represents the time when priorities switch under the optimal policy.

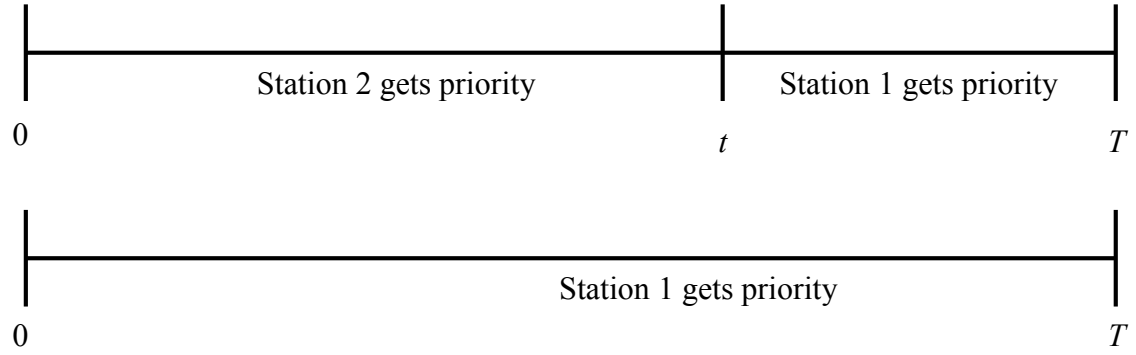
Case 6:

$$\theta_2 < \theta_1 \longrightarrow \frac{\theta_2}{\theta_1} < 1$$

$$\mu_1(1 - p_{12}) < \mu_2 \longrightarrow \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < 1$$

Again there are two options for the greedy solution.

Option 1: The greedy solution gives priority to Station 1



For this option the following condition must hold (derived from Proposition 3):

$$\mu_1(\theta_1 - \theta_2 p_{12}) > \mu_2\theta_2$$

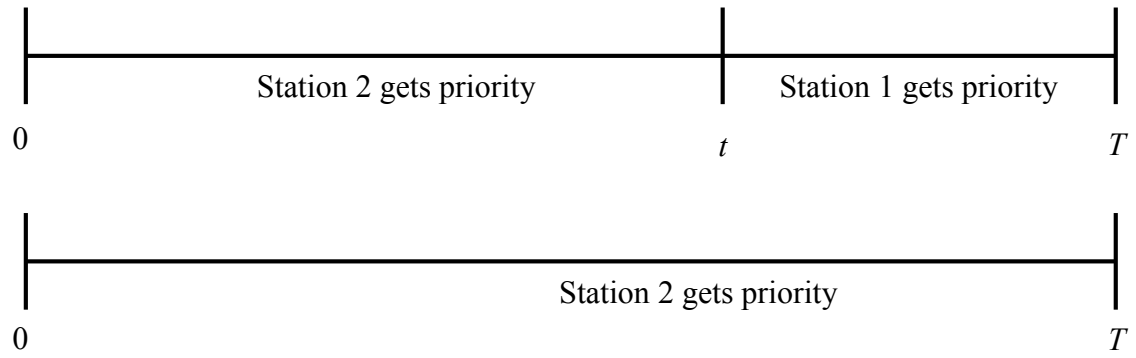
$$\frac{\theta_2}{\theta_1} < \frac{\mu_1}{\mu_2 + \mu_1 p_{12}}$$

$$\downarrow$$

$$\frac{\theta_2}{\theta_1} < \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < 1$$

We expect the largest difference between the solutions when t is as close to T as possible. This will happen when the ratio is: $\frac{\theta_2}{\theta_1} = \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} - \varepsilon$.

Option 2: The greedy solution gives priority to Station 2.



For this option the following condition must hold (derived from Proposition 3):

$$\mu_1(\theta_1 - \theta_2 p_{12}) < \mu_2\theta_2$$
$$\frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < \frac{\theta_2}{\theta_1}$$
$$\downarrow$$
$$\frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < \frac{\theta_2}{\theta_1} < 1$$

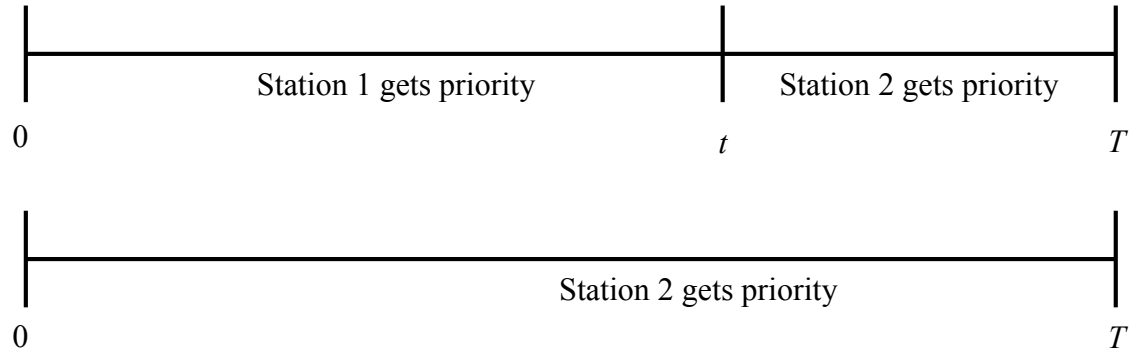
Until time *t* the two policies are identical so the largest difference between them is when *t* is as close to 0 as possible. This will happen when: $\frac{\theta_2}{\theta_1} = \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} + \varepsilon$.

Case 8:

$$\theta_2 > \theta_1 \longrightarrow \frac{\theta_2}{\theta_1} > 1$$
$$\mu_1(1 - p_{12}) > \mu_2 \longrightarrow \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} > 1$$

There are two options for the greedy solution: either Station 1 or Station 2 gets priority throughout the entire time interval.

Option 1: The greedy solution gives priority to Station 2.

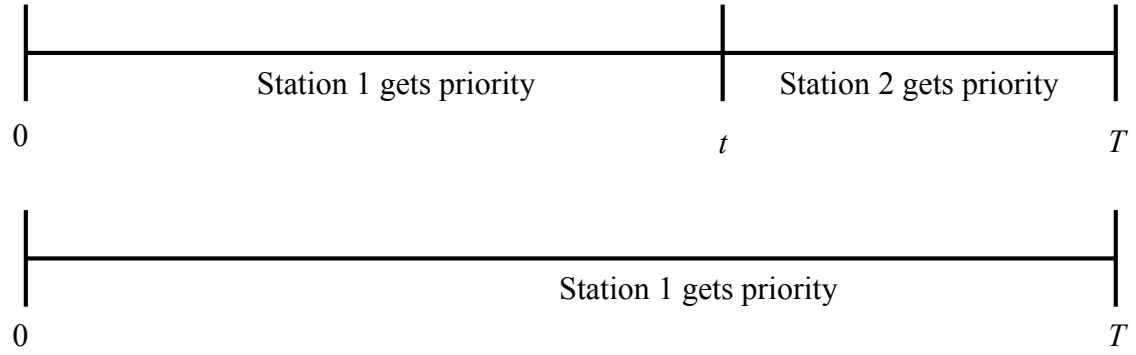


For this option the following condition must hold (derived from Proposition 3):

$$\mu_1(\theta_1 - \theta_2 p_{12}) < \mu_2\theta_2$$
$$\mu_1\theta_1 < \mu_2\theta_2 + \mu_1\theta_2 p_{12}$$
$$\mu_1\theta_1 < \theta_2(\mu_2 + \mu_1 p_{12})$$
$$\frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < \frac{\theta_2}{\theta_1}$$
$$\downarrow$$
$$1 < \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} < \frac{\theta_2}{\theta_1}.$$

The largest difference between the optimal solution and the greedy solution will be when *t*, the time point in which the priority switches in the optimal solution, will be as close to *T* as possible. This will happen when: $\frac{\theta_2}{\theta_1} = \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} + \varepsilon$ where ε is positive and small enough.

Option 2: The greedy solution gives priority to Station 1



For this option the following condition must hold (derived from Proposition 3):

$$\mu_1(\theta_1 - \theta_2 p_{12}) > \mu_2\theta_2$$
$$\frac{\mu_1}{\mu_2 + \mu_1 p_{12}} > \frac{\theta_2}{\theta_1}$$
$$\downarrow$$
$$1 < \frac{\theta_2}{\theta_1} < \frac{\mu_1}{\mu_2 + \mu_1 p_{12}}$$

Until time *t* the two policies are identical so the largest difference between the optimal solution and the greedy solution will be when *t*, the time point in which the priority switches in the optimal solution, will be as close to 0 as possible. That will happen when the ratio $\frac{\theta_2}{\theta_1} = \frac{\mu_1}{\mu_2 + \mu_1 p_{12}} - \varepsilon$.

The following table presents results for different scenarios that represent the different cases and the difference in percentage (Δ) between the optimal and greedy solutions.

Scenario 1 is when $\mu_1 = 1/30,\ \mu_2 = 1/100,\ \mathrm{p}_{12} = 0.25 \rightarrow \frac{\mu_1}{\mu_2 + \mu_1 \mathrm{p}_{12}} = 1.818$ with *N*=10 and a quadratic arrival rate $\lambda(t) = -1 \cdot 10^{-5} t^2 + 0.0044t, \quad 0 \le t \le 440$. When the arrival rate was constant $\lambda(t) = 0.8, \quad 0 \le t \le 200$ we call the scenario, Scenario 2. Scenario 3 is for Case 6 and the parameter values are $\mu_1 = 1/30,\ \mu_2 = 1/100,\ \mathrm{p}_{12} = 0.75 \rightarrow \frac{\mu_1}{\mu_2 + \mu_1 \mathrm{p}_{12}} = 0.9523$ with *N*=10 and the same constant arrival rate. Scenario 4 is identical to Scenario 3 only with the quadratic arrival rate.

No.	Case	Scenario	$\frac{\theta_2}{\theta_1}$	Δ
1	8, Option 1	1	3	4.55%
2	8, Option 1	1	2.3	7.89%
3	8, Option 1	1	1.90	10.17%
4	8, Option 2	1	1.03	0.00%
5	8, Option 2	1	1.70	0.04%
6	8, Option 2	1	1.80	0.06%
7	8, Option 2	2	1.03	0.00%
8	8, Option 2	2	1.70	0.02%
9	8, Option 2	2	1.80	0.04%
10	6, Option 1	3	0.600	0.00%
11	6, Option 1	3	0.938	1.15%
12	6, Option 1	3	0.949	1.33%
13	6, Option 1	3	0.95	1.34%
14	6, Option 2	3	0.955	0.00%
15	6, Option 2	3	0.984	0.00%
16	6, Option 2	4	0.955	0.00%

No.	Case	Scenario	$\frac{\theta_2}{\theta_1}$	Δ
17	6, Option 2	4	0.984	0.00%

Based on these experiments we expect that for Cases 6b and 8b, when the greedy solution prioritizes the same station that the optimal solution prioritizes first, the price of using the greedy policy to be very small even for the worst case. Specifically, our experiments indicated less than 0.1%. For Cases 6a and 8a the price of using the greedy solution may be higher (e.g., 10%) for the worst case. Thus for these cases our advice is to solve Problem (P3).

In order to understand the effect of the number of surgeons on the difference between the optimal and greedy policies we conducted a sensitivity analysis by solving the optimal and greedy problems for a varying number of resources, *N*. The figure below presents an example for Case 8a where we found the largest difference between the greedy and the optimal solutions. For small and large numbers of surgeons the differences between the optimal and greedy solutions were smaller when compared to the intermediate number of resources (e.g., 5 to 13). An intuitive interpretation is that when there is a severe shortage of resources (e.g., a national disaster), a reasonable policy would achieve reasonable results. When there are enough resources, then again the prioritization policy is not very important. When we draw away from these two extreme situations, the importance of using the optimal allocation policy increases.

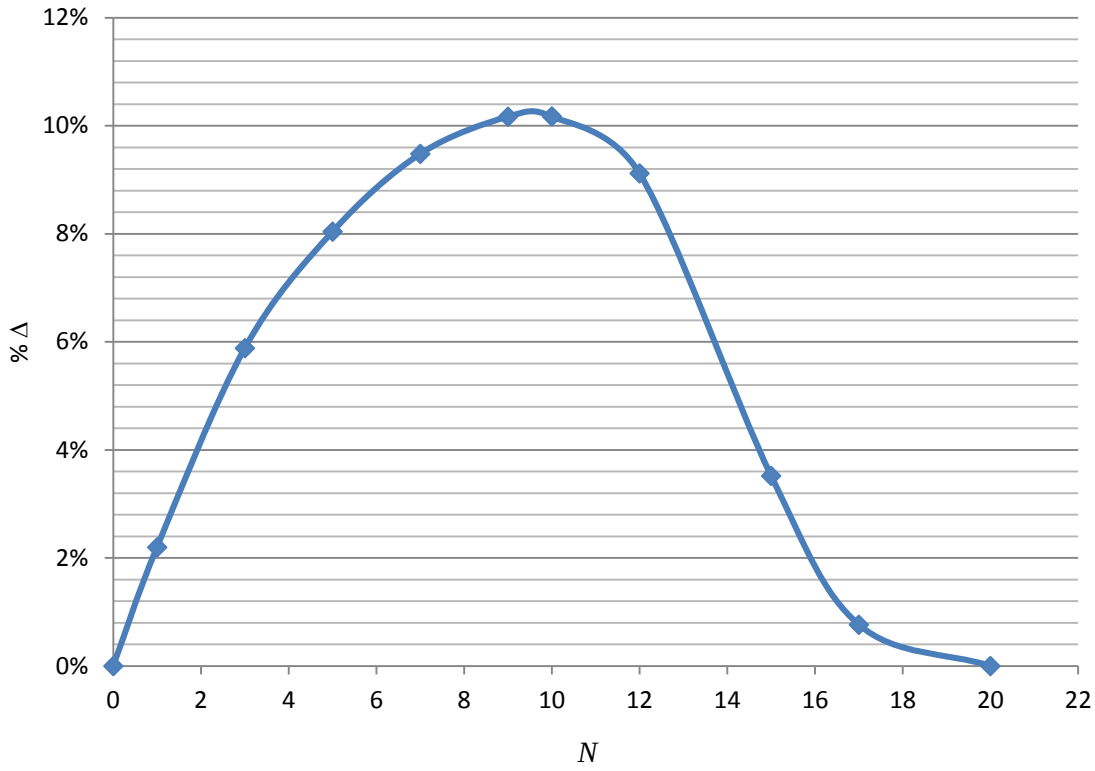


Figure: The difference (Δ) between the greedy and optimal solutions for Case 8a with parameters $\mu_1 = 1/30,\ \mu_2 = 1/100,\ \mathrm{p}_{12} = 0.25$ and the quadratic arrival rate is

$\lambda(t) = -1 \cdot 10^{-5} t^2 + 0.0044t, \quad 0 \le t \le 440$.

Proposition 6: Problem (P5) is a linear formulation of the optimization problem that satisfies the minimal time window allocation constraint.

Proof: We start with:

$$\underset{N_1(\cdot),\, N_2(\cdot)}{Min} \quad \sum_{t=0}^{T}[\theta_1 Q_1(t) + \theta_2 Q_2(t)]$$

such that for $t = 0,1,...,T-1$:

$Q_1(t+1) = (1-\theta_1)Q_1(t) + \lambda_1(t) - \mu_1(Q_1(t) \wedge N_1(t))$

$Q_2(t+1) = (1-\theta_2)Q_2(t) + p_{12}\mu_1(Q_1(t) \wedge N_1(t)) - \mu_2(Q_2(t) \wedge N_2(t))$

$N_1(t) + N_2(t) \le N$

$N_1(t),\ N_2(t),\ Q_1(t),\ Q_2(t) \ge 0$

$Q_1(0) = 0,\ \ Q_2(0) = 0$

and

$$N_i(u) = N_i(u+1) = ... = N_i(u+S-1) \qquad i = 1,2;\ \mathrm{u} = 1, S, 2S... \left\lfloor \frac{T}{S} \right\rfloor S$$

Variables $Z_1(\cdot)$ and $Z_2(\cdot)$ are defined as:

$$Z_1(t) = Q_1(t) \wedge N_1(t) \qquad \forall t \in \{0, T-1\}$$
$$Z_2(t) = Q_2(t) \wedge N_2(t) \qquad \forall t \in \{0, T-1\} \quad ,$$

which are equivalent to these linear constraints:

$$Z_1(\mathrm{t}) \le N_1(t),\ Z_1(\mathrm{t}) \le Q_1(t) \quad \forall t \in \{0, T-1\}$$
$$Z_2(\mathrm{t}) \le N_2(t),\ Z_2(\mathrm{t}) \le Q_2(t) \quad \forall t \in \{0, T-1\} \quad .$$

$Z_1(\cdot)$ and $Z_2(\cdot)$ should appear with a negative sign in the objective function to assure that the solution sets them at their maximum possible value. $Z_2(\cdot)$ always appears in the objective with a negative sign, but $Z_1(\cdot)$ appears with a negative sign only if the following condition holds:

$$-\theta_1\mu_1 + \theta_2 p_{12}\mu_1 < 0 \quad \Rightarrow \quad \theta_2 p_{12} < \theta_1 \quad .$$

This condition holds for all the cases where the mortality rate in Station 1 is higher than in Station 2 and p_{12} is smaller than 1. We expect this condition to be true for MCEs.

Substituting $Z_1(\cdot)$ and $Z_2(\cdot)$ into the problem results in:

$$\underset{N_1(\cdot),\, N_2(\cdot)}{Min} \quad \sum_{t=0}^{T}[\theta_1 Q_1(t) + \theta_2 Q_2(t)]$$

such that for $t = 0,1,...,T-1$:

$Q_1(t+1) = Q_1(t) + \lambda_1(t) - \mu_1 Z_1(t) - \theta_1 \cdot Q_1(t)$

$Q_2(t+1) = Q_2(t) + p_{12} \cdot \mu_1 Z_1(t) - \mu_2 \cdot Z_2(t) - \theta_2 \cdot Q_2(t)$

$N_1(t) + N_2(t) \le N$

$Z_i(\mathrm{t}) \le N_i(t), Z_i(\mathrm{t}) \le Q_i(t) \qquad i = 1,2$

$Z_i(\mathrm{t}) \ge 0 \qquad i = 1,2$

$N_1(t),\ N_2(t),\ Q_1(t),\ Q_2(t) \ge 0$

$Q_1(0) = Q_2(0) = 0$

and

$$N_i(u) = N_i(u+1) = ... = N_i(u+S-1) \qquad i = 1,2;\ \mathrm{u} = 0, S, 2S... \left\lfloor \frac{T}{S} \right\rfloor S$$

We can also change the summation index so that the objective includes the terms $Q_i(t+1)$:

$$\underset{N_1(\cdot),\, N_2(\cdot)}{Min} \quad \sum_{t=0}^{T-1}[\theta_1 Q_1(t+1) + \theta_2 Q_2(t+1)].$$