Priority Queues

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Model Description

M/M/N with Priorities

- N statistically identical servers;
- J customer types, indexed $1, 2, \ldots J$;
- Type i has a priority over type $j \quad \Leftrightarrow \quad i < j$;
- FCFS within each type queue;
- Non-preemptive priority (Later preemptive)

Type i: Poisson arrivals at rate λ_i
Exponential service time at rate μ, **the same for all types**

Steady-state $\Leftrightarrow \rho = \rho_1 + \ldots + \rho_J$,
where $\rho_i = \frac{\lambda_i}{N\mu}$ is fraction of time allocated to class i

$M/M/1$ with Non-Preemptive Priority

Calculation of $E(W_q^i)$ = average wait of class i.

1. $$E(W_q^1) = E(R) + \frac{1}{\mu}E(L_q^1) = \frac{\rho}{\mu} + \rho_1 E(W_q^1)$$

$$\Rightarrow \qquad E(W_q^1) = \frac{\rho}{\mu(1-\rho_1)}$$

2. $$E(W_q^2) = E(R) + \underbrace{\frac{1}{\mu}E(L_q^1) + \frac{1}{\mu}E(L_q^2)}_{due\ to\ type\ 1\ and\ 2\ in\ queue} + \underbrace{\frac{1}{\mu}\lambda_1 E(W_q^2)}_{due\ to\ new\ type\ 1}$$

$$= \frac{\rho}{\mu} + \rho_1 E(W_q^1) + \rho_2 E(W_q^2) + \rho_1 E(W_q^2)$$

$$\Rightarrow \qquad E(W_q^2) = \frac{\rho/\mu + \rho_1 E(W_q^1)}{1-\rho_1-\rho_2} = \frac{\rho}{\mu(1-\rho_1)(1-\rho_1-\rho_2)}$$

⋮

i. $$E(W_q^i) = E(R) + \frac{1}{\mu}\left(E(L_q^1) + \ldots E(L_q^i)\right) + \frac{1}{\mu}\sum_{k=1}^{i-1}\lambda_k E W_q^i$$

$$\Rightarrow \qquad E(W_q^i) = \frac{\rho/\mu + \sum_{k=1}^{i-1}\rho_k E(W_q^k)}{1-\rho_1-\rho_2-\ldots-\rho_i}$$

$$= \frac{\rho/\mu}{(1-\rho_1-\ldots-\rho_{i-1})(1-\rho_1-\ldots-\rho_i)}$$

Convenient notation: $\sum_{k=1}^{i}\rho_k = \rho_{\leq i}, \quad \forall 1 \leq i \leq J$

M/M/1: A Numerical Example

$M/M/1$

Two types, $i = 1, 2,\quad \mu = 10\ \frac{customers}{minute}$,

$\lambda_1 = 4\ \frac{customers}{minute}$, $\lambda_2 = 3\ \frac{customers}{minute}$

No priorities:

$$E(W_q^1) = E(W_q^2) = E(W_q) = \frac{\rho}{\mu(1-\rho_1)} = 14\ \ sec.$$

Non-Preemptive Priority:

$$E(W_q^1) = \frac{\rho}{\mu(1-\rho_1)} = 7\ \ sec$$

$$E(W_q^2) = \frac{\rho}{\mu(1-\rho_1)(1-\rho_1-\rho_2)} = 23.32\ \ sec$$

Note:

$$\frac{E(W_q^1)}{E(W_q^2)} = 1 - \rho$$

Non-Preemptive Priority: $E(W_q^i)$

Recall: $E_{2,N} \equiv P(Wait > 0)$ in a single class M/M/N system (Erlang-C formula)

$$E_{2,N} = \frac{(N\rho)^N}{N!(1-\rho)} \left[\sum_{k=0}^{N-1} \frac{(N\rho)^k}{k!} + \frac{(N\rho)^N}{N!(1-\rho)} \right]^{-1} \tag{1}$$

<u>Non-Preemptive</u> (Kella and Yechiali 1985)

The average waiting time of the i^{th} class is:

$$E(W_q^i) = \frac{1}{N\mu} \frac{E_{2,N}}{(1-\rho_{\leq i-1})(1-\rho_{\leq i})}, \tag{2}$$

where $\rho_{\leq i} \equiv \sum_{k=1}^{i} \rho_k$

Proof:

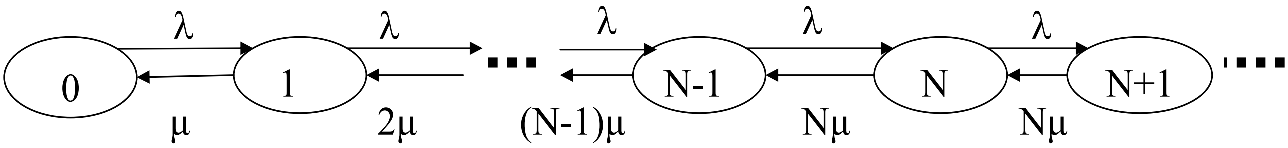
Two steps:

1. $P(W_q^i > 0) = E_{2,N}, \quad \forall i = 1, \dots K$

2. $E(W_q^i | W_q^i > 0) = \dfrac{1}{N\mu} \dfrac{1}{(1-\rho_{\leq i-1})(1-\rho_{\leq i})}$

Non-Preemptive Priority: $E(W_q^i)$. Proof

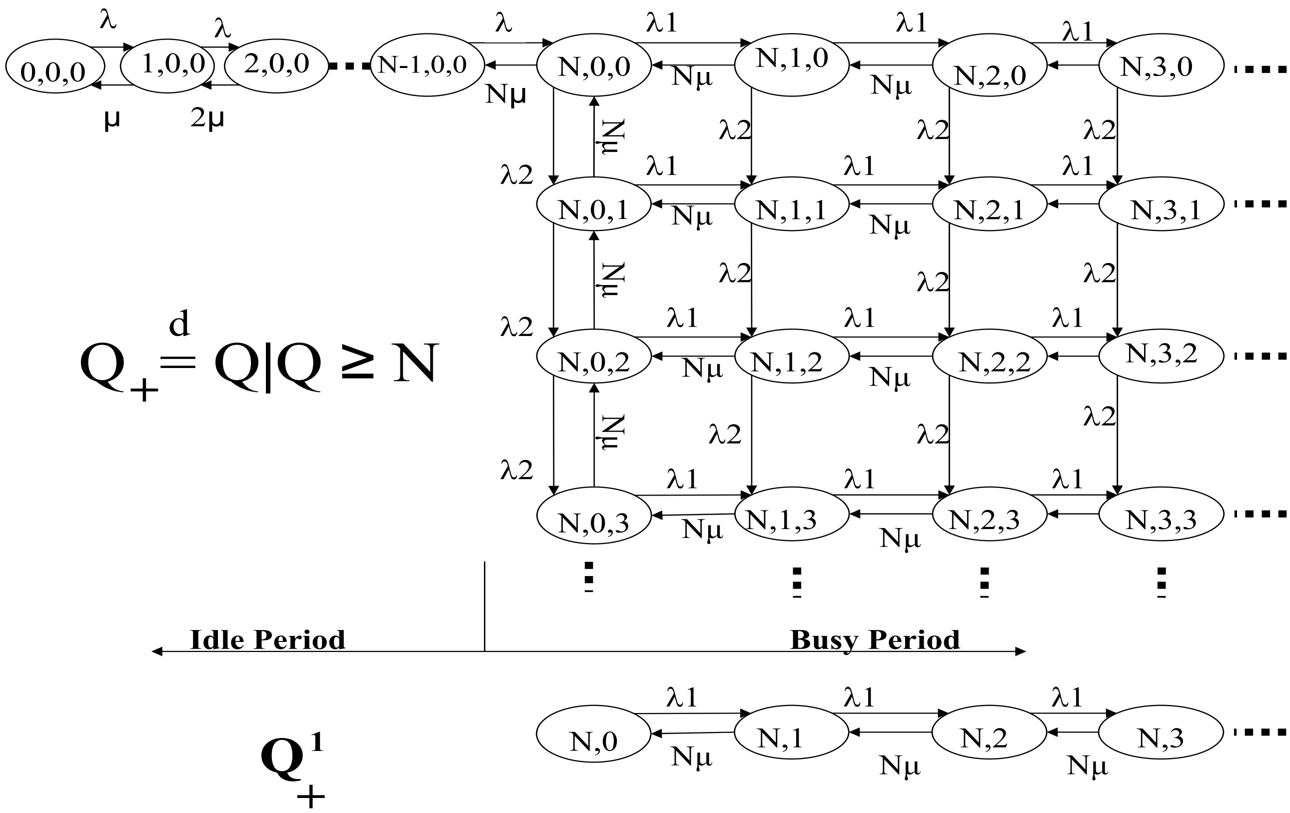
Step 1



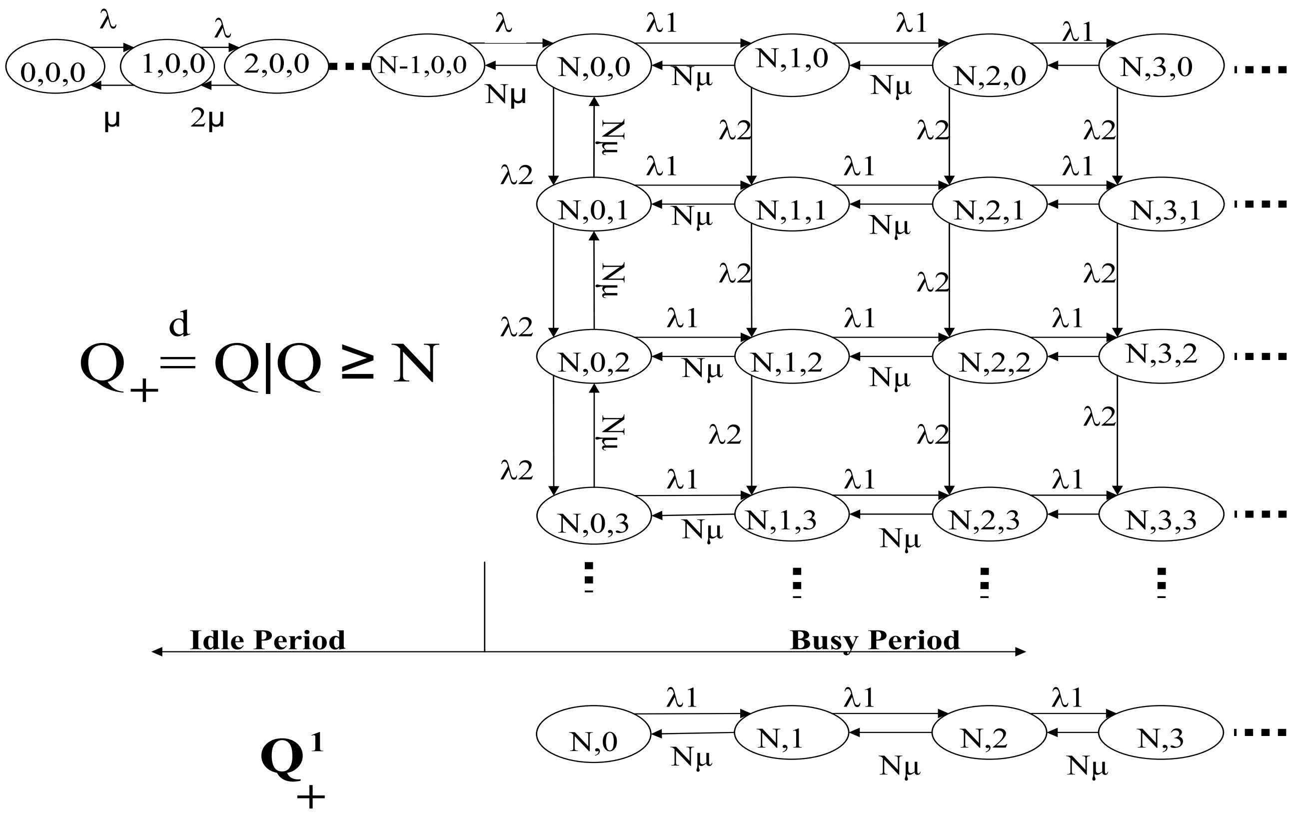
$Q =$ total number in system at time $t \geq 0$, $Q \overset{d}{=} M/M/N, \quad \lambda, \mu$

$\Rightarrow \quad P(W_q^i > 0) = P_{M/M/N}(Wait > 0) = E_{2,N}$

Step 2



$E(W_q^i)$: Proof Continued



Define: $Q_+^i \equiv$ total number of delayed customers of types $1, \ldots, i$

Then $\quad Q_+^1 \equiv$ number of the delayed type 1 customers

$$Q_+^1 \stackrel{d}{=} M/M/1, \quad \lambda_1, N\mu$$

$$\Rightarrow \qquad (W_q^1 | W_q^1 > 0) \stackrel{d}{=} \exp(mean = \frac{1}{N\mu(1-\rho_1)})$$

$$E(W_q^1) = \frac{E_{2,N}}{N\mu(1-\rho_1)}$$

$E(W_q^i)$. Proof Continued

Type i customer:

Given wait, he ”sees” only types $1, \ldots, i$.

$M/G/1$ queue,

where $G =$ Busy Period of M/M/1 with $\lambda_{\le i-1} = \sum_{k=1}^{i-1} \lambda_k$

Each service completion lasts on average $\frac{1}{N\mu}$

Mean number of arrivals $1, \ldots, i-1$ during each Busy-Period is $\dfrac{1}{(1-\rho_{\le i-1})}$

Queue length upon type i arrival: $\quad L_q^{\le i} \stackrel{d}{=} Geom_0(1-\rho_{\le i})$

$\Rightarrow \quad$ # busy periods $\stackrel{d}{=} L^{\le i} \stackrel{d}{=} Geom(1-\rho_{\le i})$

where $L^{\le i} \equiv$ total number of customers in $M/M/1$, $\lambda^{\le i}$, $N\mu$.

Thus,

$$
\begin{aligned}
E(W_q^i | W_q^i > 0) &= E(L_q^{\le i}) \times \underbrace{\frac{1}{N\mu(1-\rho_{\le i-1})}}_{Busy-Period\ \ Duration} \\
&= \frac{1}{N\mu(1-\rho_{\le i-1})(1-\rho_{\le i})}
\end{aligned}
$$

Conclude:

$$
E(W_q^i) = \frac{E_{2,N}}{N\mu(1-\rho_{\le i-1})(1-\rho_{\le i})}
$$

Preemptive Priority

Recursive relation:

$$E(W_q^i) = [\lambda_{\le i}\overline{W}^{(1\to i)} - \sum_{j=1}^{i-1}(\lambda_j E(W_q^j))]/\lambda_i$$

where $\overline{W}^{(1\to i)} \equiv E(W_q)$ in a single-class M/M/N queue

with $\lambda_{\le i} = \sum_{j=1}^{1} \lambda_i$ (i.e. ignoring lower types $i+1,\ldots,J$).

Proof:

Define:

$L_q^i \equiv$ average number of class i customers in queue.

$L^{(1\to i)} \equiv$ average number of customers in a single-class M/M/N queue with arrival rate $\lambda_{\le i}$

$$L^{(1\to i)} = \sum_{j=1}^{i} L_q^i \quad \text{(does not depend on queueing policy)}$$

Calculation of $E(W_q^i) =$ average wait of class i:

1. $E(W_q^1) = \overline{W}^{(1\to 1)}$

2. $L^{(1\to 2)} = L_1 + L_2$

$\Rightarrow \quad (\lambda_1+\lambda_2)\overline{W}^{(1\to 2)} = \lambda_1 E(W_q^1) + \lambda_2 E(W_q^2)$

$\Rightarrow \quad E(W_q^2) = [\lambda_{\le 2}\overline{W}^{(1\to 2)} - \lambda_1 E(W_q^1)]/\lambda_2$

$\vdots$

i. $$E(W_q^i) = [\lambda_{\le i}\overline{W}^{(1\to i)} - \sum_{j=1}^{i-1}(\lambda_j E(W_q^j))]/\lambda_i$$

QED and ED:

Preemptive vs. Non-Preemptive

QED: $N \sim R + \beta\sqrt{R}$ for some β, $0 < \beta < \infty$

$$\Leftrightarrow \quad \rho \sim 1 - \frac{\beta}{\sqrt{N}}\,, \quad \text{namely} \quad \lim_{N\to\infty} \sqrt{N}\,(1-\rho_N) = \beta.$$

ED: $\Leftrightarrow \quad \lim_{N\to\infty} N\,(1-\rho_N) = \gamma$ for some γ, $0 < \gamma < \infty$

Theorem

(Halfin-Whitt, 1981) QED $\Leftrightarrow \quad \lim_{N\to\infty} E_{2,N} = \alpha \equiv \left[1 + \frac{\beta}{h(-\beta)}\right]^{-1}.$

Example: Two Customer Types

$$E_{np}(W_q^1) = \frac{P_\lambda(Wait > 0)}{N\mu(1-\rho_1)} \qquad E_{np}(W_q^2) = \frac{P_\lambda(Wait > 0)}{N\mu(1-\rho_1)(1-\rho)}$$

$$E_p(W_q^1) = \frac{P_{\lambda 1}(Wait > 0)}{N\mu(1-\rho_1)} \qquad E_p(W_q^2) = \frac{\lambda\frac{P_\lambda(Wait>0)}{N\mu(1-\rho)} - \lambda_1\frac{P_{\lambda 1}(Wait>0)}{N\mu(1-\rho_1)}}{\lambda_2}$$

Physics of Low Priority: as $N \uparrow \infty$, $P(Service\ \ Interruption) \downarrow 0$

QED: Preemptive vs. Non-Preemptive Cont.

$P_\lambda(Wait > 0) \approx \alpha$

$\Rightarrow \quad E_{np}(W_q^1)$ converges to 0 at rate $\frac{1}{N}$

$$\lim_{N\to\infty} \frac{1}{N\mu(1-\rho_1)(1-\rho)} = \lim_{N\to\infty} \frac{1}{\sqrt{N}\beta\rho_2}$$

$\Rightarrow \quad E_{np}(W_q^2)$ converges to 0 at rate $\dfrac{1}{\sqrt{N}}$

$$P_{\lambda_1}(Wait) = \frac{1}{\sqrt{2\pi N}} \cdot \frac{e^{\rho_2}}{1-\rho_1} \cdot \rho_1^N + o(1/N)$$

$\Rightarrow \quad E_{pr}(W_q^1)$ converges to 0 at rate $\dfrac{\rho_1^N}{N\sqrt{N}}$

$P_\lambda(Wait > 0) \approx \alpha$

$$\lim_{N\to\infty} \frac{\lambda}{\lambda_2 N\mu(1-\rho)} = \lim_{N\to\infty} \frac{1}{\sqrt{N}\rho_2\beta}$$

$\Rightarrow \quad E_{np}(W_q^2)$ converges to 0 at rate $\frac{1}{\sqrt{N}}$

ED: Preemptive vs. Non-Preemptive Cont.

$P_\lambda(Wait > 0) \approx 1$

$\Rightarrow \qquad E_{np}(W_q^1)$ converges to 0 at rate $\frac{1}{N}$

$$\lim_{N\to\infty} \frac{1}{N\mu(1-\rho_1)(1-\rho)} = \lim_{N\to\infty} \frac{1}{\gamma\rho_2}$$

$\Rightarrow \qquad E_{np}(W_q^2)$ converges to 0 at rate 1

$$P_{\lambda_1}(Wait) = \frac{1}{\sqrt{2\pi N}} \cdot \frac{e^{\rho_2}}{1-\rho_1} \cdot \rho_1^N + o(1/N)$$

$\Rightarrow \qquad E_{pr}(W_q^1)$ converges to 0 at rate $\dfrac{\rho_1^N}{N\sqrt{N}}$

$P_\lambda(Wait > 0) \approx 1$

$$\lim_{N\to\infty} \frac{\lambda}{\lambda_2 N\mu(1-\rho)} = \lim_{N\to\infty} \frac{1}{\rho_2\gamma}$$

$\Rightarrow \qquad E_{np}(W_q^2)$ converges to 0 at rate 1.

QED and ED:

Summary of Results

Expected Waiting Time: Convergence Rate

	QED		ED	
	N-Pr	Pr	N-Pr	Pr
I	$\Theta(1/N)$	$\Theta(\frac{\rho_1^N}{N\sqrt{N}})$	$\Theta(1/N)$	$\Theta(\frac{\rho_1^N}{N\sqrt{N}})$
II	$\Theta(1/\sqrt{N})$	$\Theta(1/\sqrt{N})$	$\Theta(1)$	$\Theta(1)$

<u>QED</u>

$$E_{np}(W_q^1)\cdot N \quad \to \frac{\alpha}{\rho_2\mu}$$

$$E_{np}(W_q^2)\cdot\sqrt{N} \quad \to \frac{\alpha}{\rho_2\mu\beta}$$

$$E_{pr}(W_q^1)\cdot\frac{N\sqrt{N}}{(\rho_1)^N} \quad \to \frac{e^{\rho_2}}{\sqrt{2\pi\rho_2^2}\mu}$$

$$E_{pr}(W_q^2)\cdot\sqrt{N} \quad \to \frac{\alpha}{\rho_2\beta\mu}$$

<u>ED</u>

$$E_{np}(W_q^1)\cdot N \quad \to \frac{1}{\rho_2\mu}$$

$$E_{np}(W_q^2) \quad \to \frac{1}{\rho_2\mu\gamma}$$

$$E_{pr}(W_q^1)\cdot\frac{N\sqrt{N}}{(\rho_1)^N} \quad \to \frac{e^{\rho_2}}{\sqrt{2\pi\rho_2^2}\mu}$$

$$E_{pr}(W_q^2) \quad \to \frac{1}{\rho_2\mu\gamma}$$

Non-Preemptive Priority + Abandonment

M/M/N+M with Priorities

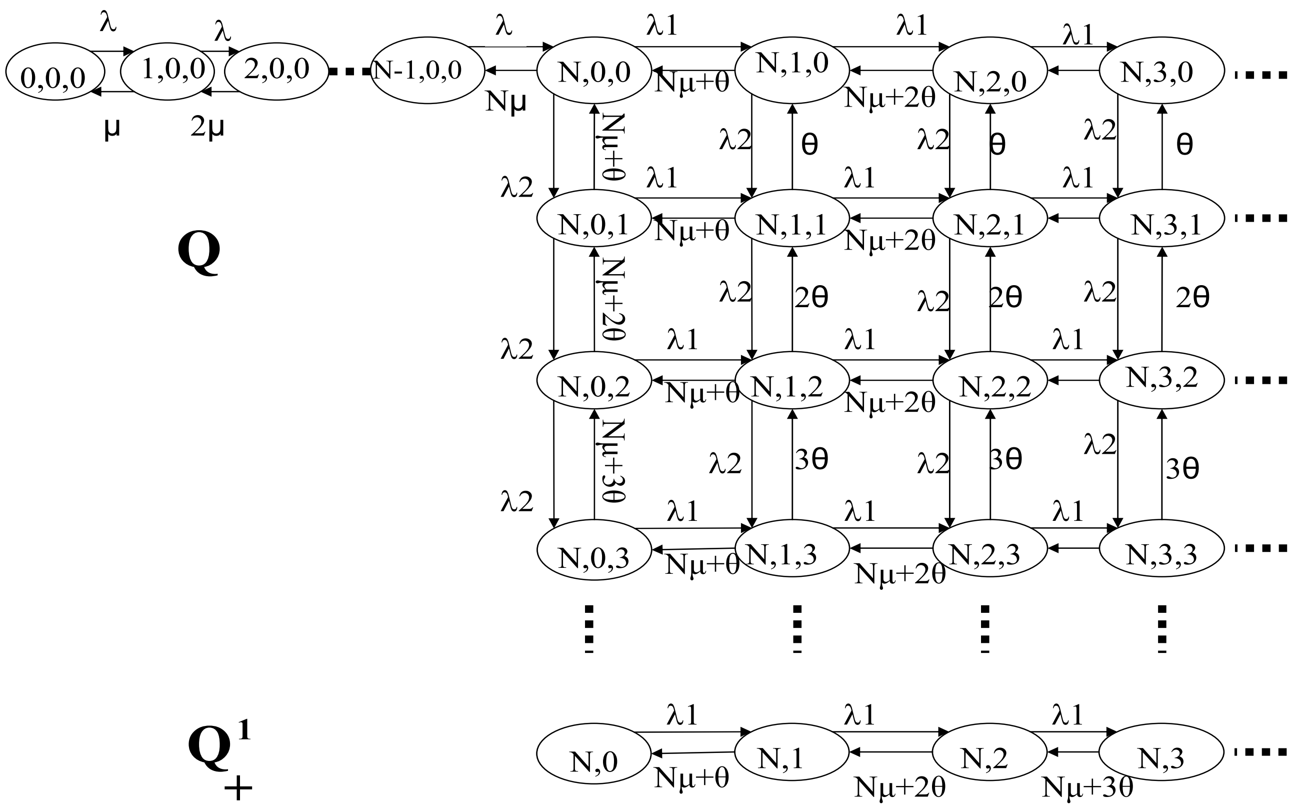
- N statistically identical servers;
- J customer types, indexed $1, 2, \ldots J$;
- Type i has a priority over type $j \quad \Leftrightarrow \quad i < j$;
- FCFS within each type queue;
- Non-preemptive priority (Later preemptive)

Type i: Poisson arrivals at rate λ_i
Exponential service time at rate μ
Exponential patience with rate θ

Steady-state always exists;

$\rho = \rho_1 + \ldots + \rho_J$,
where $\rho_i = \frac{\lambda_i}{N\mu}$ is offered load per server allocated to class i

Non-Preemptive Priority: $P(Wait > 0)$



$$E(W_q^1) = P(Wait > 0)E(W_q^1 | W_q^1 > 0)$$

$P(W_q^i > 0) =$Erlang-A $(M/M/N + M, \quad \lambda, \quad \mu, \quad \theta)$

(Delay probability does not depend on the queueing discipline)

Non-Preemptive Priority: $E(W_q^1 | W_q^1 > 0)$

Recall :

$$Q_+^i \equiv \text{number of type } i \text{ customers}$$

Then

$$Q_+^1 \stackrel{d}{=} M/M/1 + M, \quad \lambda_1, \quad N\mu + \theta, \quad \theta$$

$$\Rightarrow \qquad E(W_q^1 | W_q^1 > 0) = E_{\lambda_1}(W | W > 0) = \tfrac{1}{\theta} P_{\lambda_1}(Aband | W > 0),$$

where

$E_{\lambda_1}(W|W > 0)$, $P_{\lambda_1}(Aband|W > 0)$ are calculated

in $M/M/1 + M$ $(\lambda_1, N\mu + \theta, \theta)$

Example: $J = 2$

Define: $L^{(1 \to 2)} \equiv$ expected total number of delayed customers

$$L^{(1\to 2)} = L_q^1 + L_q^2$$

Little's Law:

$$E(W^{(1\to 2)}) = \frac{\lambda_1}{\lambda} E(W_q^1) + \frac{\lambda_2}{\lambda} E(W_q^2)$$

$$\Rightarrow \qquad E(W_q^2) = \frac{\lambda E(W_{tot}) - \lambda_1 E(W_q^1)}{\lambda_2}$$

Non-Preemptive Priority: J Types

The Algorithm:

Step 1: $E_{np}(W_q^1) = P_\lambda(Wait > 0) \cdot E_{\lambda_1}(W|W > 0)$

Step 2: "Merge" the first two types a single type with $\lambda_{\leq 2} = \lambda_1 + \lambda_2$

$$E_{np}(W_q^{2\to 1}) = P_\lambda(Wait > 0) \cdot E_{\lambda_{\leq 2}}(W|W > 0).$$

$$E_{np}(W_q^{2\to 1}) = \frac{\lambda_1}{\lambda_{\leq 2}} E_{np}(W_q^1) + \frac{\lambda_2}{\lambda_{\leq 2}} E_{np}(W_q^2)$$

$$\Rightarrow \qquad E_{np}(W_q^2) = \frac{\lambda_{\leq 2} \cdot E_{np}(W_q^{2\to 1}) - \lambda_1 E_{np}(W_q^1)}{\lambda_2}.$$

In general:

repeating Steps 2 and 3 for $i \leq J - 1$ we get the following relation:

$$E_{np}(W_q^{(i+1)\to 1}) = \frac{\lambda_{\leq i}}{\lambda_{\leq i+1}} E_{np}(W_q^{i\to 1}) + \frac{\lambda_{i+1}}{\lambda_{\leq i+1}} E_{np}(W_q^{i+1})$$

$$\Rightarrow E_{np}(W_q^{i+1}) = \frac{\lambda_{\leq i+1} \cdot E_{np}(W_q^{(i+1)\to 1}) - \lambda_{\leq i} E_{np}(W_q^{i\to 1})}{\lambda_{i+1}}.$$

QED with Abandonments:

Preemptive vs. Non-Preemptive

QED: $N \sim R + \beta\sqrt{R}$ for some β, $-\infty < \beta < \infty$

$$\Leftrightarrow \quad \rho \sim 1 - \frac{\beta}{\sqrt{N}}, \quad \text{namely} \quad \lim_{N\to\infty} \sqrt{N}\,(1-\rho_N) = \beta.$$

Theorem

(Garnett-Mandelbaum-Reiman, 2002)

$$\text{QED} \Leftrightarrow \quad \lim_{N\to\infty} P(Wait > 0) = \alpha \equiv \left[1 + \frac{h(\delta)/\delta}{h(-\beta)/\beta}\right]^{-1}, \ \delta = \beta\sqrt{\mu/\theta}.$$

Example: Two Customer Types

$$E_{np}(W_q^1) = \frac{P_\lambda(Wait > 0)}{\theta} P_{\lambda_1}(Aband|Wait > 0)$$

$$E_{np}(W_q^2) = \frac{P_\lambda(Wait > 0)}{\theta} \cdot \frac{\lambda P_\lambda(Aband|Wait > 0)}{\lambda_2} - \frac{P_\lambda(Wait > 0)}{\theta} \cdot \frac{\lambda_1 P_{\lambda_1}(Aband|Wait > 0)}{\lambda_2}$$

$$E_p(W_q^1) = \frac{1}{\theta} P_{\lambda_1}(Aband)$$

$$E_p(W_q^2) = \frac{1}{\theta} \frac{\lambda P_\lambda(Aband) - \lambda_1 P_{\lambda_1}(Aband)}{\lambda_2}$$

QED with Abandonments:

Preemptive vs. Non-Preemptive Cont.

$$P_\lambda(Wait > 0) \approx \alpha$$

$$P_{\lambda_1}(Aband|Wait > 0) = \frac{1}{N} \cdot \frac{1}{1-\rho_1} \cdot \frac{\theta}{\mu} + o(1/N)$$

$\Rightarrow \qquad E_{np}(W_q^1)$ converges to 0 at rate $\frac{1}{N}$

$$P_\lambda(Aband|Wait > 0) = \frac{1}{\sqrt{N}} \cdot \sqrt{\frac{\theta}{\mu}} \cdot [h(\delta) - \delta] + o(1/N)$$

$\Rightarrow \qquad E_{np}(W_q^2)$ converges to 0 at rate $\frac{1}{\sqrt{N}}$

$$P_{\lambda_1}(Wait) = \frac{1}{\sqrt{2\pi N}} \cdot \frac{e^{\rho_2}}{1-\rho_1} \cdot \rho_1^N + o(1/N)$$

$$P_{\lambda_1}(Aband|Wait > 0) = \frac{1}{N} \cdot \frac{1}{1-\rho_1} \cdot \frac{\theta}{\mu} + o(1/N)$$

$\Rightarrow \qquad E_{pr}(W_q^1)$ converges to 0 at rate $\frac{\rho_1^N}{N\sqrt{N}}$

$$P_\lambda(Aband) = \frac{\alpha}{\sqrt{N}} \cdot \sqrt{\frac{\theta}{\mu}} \cdot [h(\delta) - \delta] + o(1/N)$$

$\Rightarrow \qquad E_{pr}(W_q^2)$ converges to 0 at rate $\frac{1}{\sqrt{N}}$

ED with Abandonments:

Preemptive vs. Non-Preemptive

ED: $N \sim R\,(1-\gamma)$ for some γ, $0 < \gamma < \infty$

$$\Leftrightarrow \quad \rho \sim 1 - \frac{1}{1-\gamma}\,, \quad \text{and} \quad \gamma = \frac{\rho - 1}{\rho}$$

Example: Two Customer Types

$$P_\lambda(Wait > 0) \approx 1$$

$$P_{\lambda_1}(Aband|Wait > 0) = \frac{1}{N} \cdot \frac{1}{1-\rho_1} \cdot \frac{\theta}{\mu} + o(1/N)$$

$\Rightarrow$ $E_{np}(W_q^1)$ converges to 0 at rate $\frac{1}{N}$

$$P_\lambda(Aband) \sim \gamma$$

$\Rightarrow$ $E_{np}(W_q^2)$ converges to 0 at rate 1

$$P_{\lambda_1}(Wait) = \frac{1}{\sqrt{2\pi N}} \cdot \frac{e^{\rho_2}}{1-\rho_1} \cdot \rho_1^N + o(1/N)$$

$$P_{\lambda_1}(Aband|Wait > 0) = \frac{1}{N} \cdot \frac{1}{1-\rho_1} \cdot \frac{\theta}{\mu} + o(1/N)$$

$\Rightarrow$ $E_{pr}(W_q^1)$ converges to 0 at rate $\dfrac{\rho_1^N}{N\sqrt{N}}$

$$P_\lambda(Aband) \sim \gamma$$

$\Rightarrow$ $E_{pr}(W_q^2)$ converges to 0 at rate 1

QED and ED with Abandonments:Summary of Results

Expected Waiting Time: Convergence Rate

	QED		ED	
	N-Pr	Pr	N-Pr	Pr
I	$\Theta(1/N)$	$\Theta(\frac{\rho_1^N}{N\sqrt{N}})$	$\Theta(1/N)$	$\Theta(\frac{\rho_1^N}{N\sqrt{N}})$
II	$\Theta(1/\sqrt{N})$	$\Theta(1/\sqrt{N})$	$\Theta(1)$	$\Theta(1)$

QED

$$E_{np}(W_q^1) \cdot N \quad \to \frac{\alpha}{\rho_2 \mu}$$

$$E_{np}(W_q^2) \cdot \sqrt{N} \quad \to \frac{\alpha}{\sqrt{\theta\mu}}[h(\delta) - \delta]$$

$$E_{pr}(W_q^1) \cdot \frac{N\sqrt{N}}{\rho_1^N} \quad \to \frac{e^{\rho_2}}{\sqrt{2\pi\rho_2^2}} \frac{\theta}{\mu}$$

$$E_{pr}(W_q^2) \cdot \sqrt{N} \quad \to \frac{\alpha}{\sqrt{\theta\mu}}[h(\delta) - \delta]$$

ED

$$E_{np}(W_q^1) \cdot N \quad \to \frac{1}{\mu\rho_2}$$

$$E_{np}(W_q^2) \quad \to \frac{1}{\theta\rho_2} \frac{\gamma}{(1-\gamma)}$$

$$E_{pr}(W_q^1) \cdot \frac{N\sqrt{N}}{\rho_1^N} \quad \to \frac{e^{\rho_2}}{\sqrt{2\pi\rho_2^2}} \frac{\theta}{\mu}$$

$$E_{pr}(W_q^2) \quad \to \frac{1}{\theta\rho_2} \frac{\gamma}{(1-\gamma)}$$