# Markovian N-Server Queues

# (Birth & Death)

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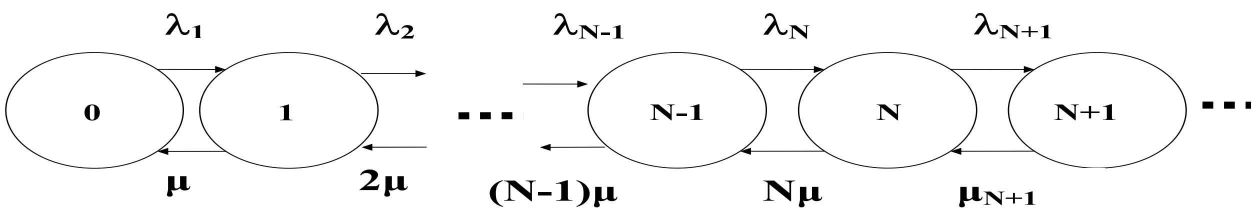



| | |
|---|---|
| Arrivals | Poisson $(\lambda)$ ; |
| Services | exp$(\mu)$ $(E(S) = 1/\mu)$ ; |
| Servers | $N$ **statistically identical**, serving FCFS ; |
| Offered load | $R = \lambda \times E(S) = \lambda/\mu$ Erlangs; |
| | $\rho = R/N$ Erlangs per server. |

$Q(t) =$ number in system (served + queued) at time $t$.

$W(k) =$ queueing time of $k$-th arrival.

Steady State: $Q(\infty)$, $W(\infty)$, when exists.

Non-idling $\Rightarrow$ $[Q(t) - N]^+$ = queue-length, and

$[Q)(t) - N]^-$ = number of idle servers.

# Examples: M/M/N, Blocking, Abandonment

M/M/$\infty$: $\lambda_k \equiv \lambda, \quad \mu_k = k \cdot \mu \quad , \quad k \geq 1.$

Steady state $\pi_k = e^{-R} R^k / k!$ Poisson $(R)$

M/M/N/N: $\lambda_k \equiv 0, \quad k \geq N.$

Steady state $$\pi_k = \frac{R^k}{k!} \Big/ \sum_{n=0}^{N} \frac{R^n}{n!} \quad , \quad 0 \leq k \leq N;$$

Erlang-B $P\{\text{Blocked}\} \doteq \boldsymbol{E_{1,N}} = \pi_N$ , by PASTA

M/M/N: $\lambda_k \equiv \lambda, \quad \mu_k = (k \wedge N) \cdot \mu \quad , \quad k \geq 1.$

Steady state $\Leftrightarrow \rho = \dfrac{\lambda}{N\mu} < 1$ , servers' utilization.

Erlang-C $$P\{\text{Wait} > 0\} \doteq \boldsymbol{E_{2,N}} = \sum_{k \geq N} \pi_k$$

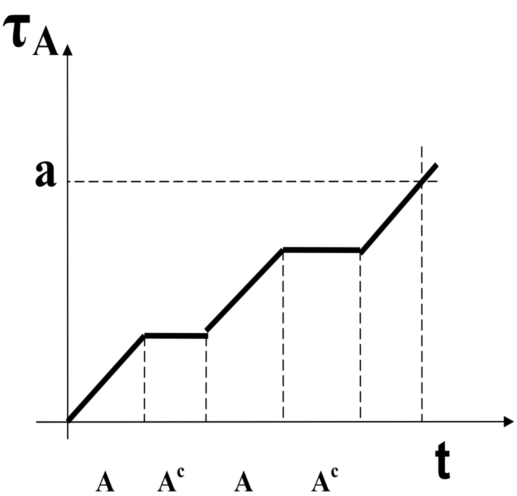
$$W(\infty) \mid W(\infty) > 0 \overset{d}{=} \exp\left(\text{mean} = \frac{1}{N\mu(1-\rho)}\right)$$

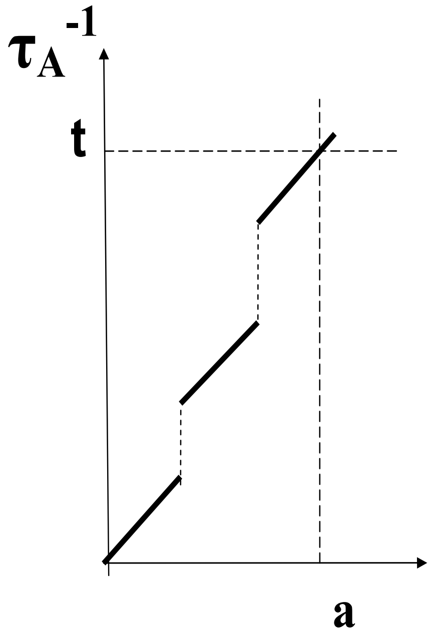
M/M/N+M: $\lambda_k \equiv \lambda, \quad \mu_k = (k \wedge N) \cdot \mu + (n-k)^+ \theta \quad , \quad k \geq 0.$

# Restriction to a Set via Time-Change

$X$ Markov , $X(\infty) \stackrel{d}{=} \pi$

$\tau_A(t) = \int_0^t 1_{\{X(u)\in A\}} du$ time in $A$



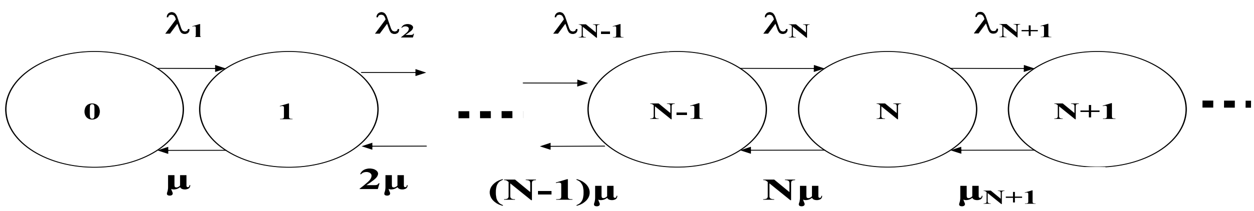

$X_A(t) = X\left(\tau_A^{-1}(t)\right)$ $X$ restricted to $A$: Markov

$X_A(\infty) \stackrel{d}{=} X(\infty)|X(\infty) \in A$

**Example**: M/M/N/N $\stackrel{d}{=}$ M/M/$\infty$ restricted to $\{0, 1, \ldots, N\}$.

$$\begin{aligned} E_{1,N} &= Pr\{X_R = N\}/Pr\{X_R \le N\}, \\ X_R &\stackrel{d}{=} \text{Poisson } (R) \end{aligned}$$

# M/M/N/N (Erlang-B): Restricted M/M/$\infty$




$M/M/\infty$:

$Q(t) =$ number in system at time $t \geq 0$, $Q(\infty) \stackrel{d}{=} \pi$

Then

$Q_- = Q$ restricted to $\{0, 1, \ldots, N\}$ : M/M/N/N ; $(\lambda, \mu)$.

$$P(Q_-(\infty) = N) = \frac{\pi(N)}{\sum_{i=0}^{N} \pi(i)} = E_{1,N}$$

**Proof:** $Q_-$ is a Markov Process:

Sojourn time in state $i \in \{0, 1, \ldots, N-1\}$ is $\exp(\lambda + i\mu)$.

State $N$: Each visit lasts $\exp(r = \lambda + N\mu)$

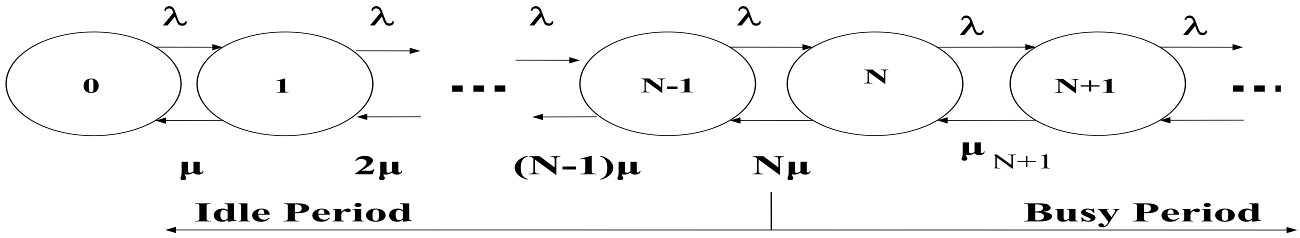
Number of visits is $Geo(p = \frac{N\mu}{\lambda + N\mu})$

Fact: Geometric$(p)$ sum of i.i.d. exponentials $(r) \stackrel{d}{=} \exp(p \cdot r)$

$\Rightarrow$ Sojourn time in state $N \stackrel{d}{=} \exp(N\mu)$.

Conclude: $Q_- \stackrel{d}{=} M/M/N/N$.

# Up/Down Crossings




Define: Idle Period

$T_{N-1,N}$ = E $\left[\, 1^{st}\right.$ hitting time of $N|Q(0) = N-1\left.\right]$.

Then
$$T_{N-1,N} = \frac{\sum_{i=0}^{N-1} \pi_i}{\lambda_{N-1}\pi_{N-1}} = \frac{1}{\lambda \pi_-(N-1)},$$

where $\pi_-$ is the distribution of the restricted $Q_-$.

Similarly: Busy Period

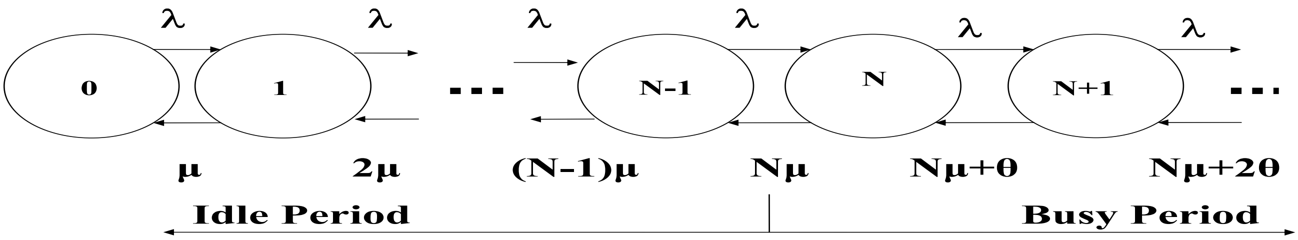
$T_{N,N-1}$ = E $\left[\, 1^{st}\right.$ hitting time of $N-1|Q(0) = N\left.\right]$.

**Proof**:

Number of Idle Excursions $\stackrel{d}{=}$ $Geometric_{\geq 0}\left(\frac{\lambda_{N-1}}{\lambda_{N-1}+\mu_{N-1}}\right)$

$$T_{N-1,N} = \underbrace{\frac{1}{\pi_-(N-1)\mu_{N-1}}}_{E(Idle\ Excursion)} \times \underbrace{\frac{\mu_{N-1}}{\lambda_{N-1}}}_{E(\#\ of\ Excursions)}$$

# M/M/N+M (Erlang-A)




$Q(t) =$ number in system at time $t \geq 0$

$Q_- = Q$ restricted to $\{0, 1, \ldots, N-1\}$ : M/M/N-1/N-1 ; $\lambda, \mu$.

$Q_+ = Q$ restricted to $\{N, N+1, \ldots\}$ : M/M/1+M ; $\lambda, N\mu + \theta, \theta$.

Evolution of $Q$: Alternates between M/M/1+M ($Q_+$) and M/M/N-1/N-1 ($Q_-$).

$$P(\text{Wait} > 0) = \frac{T_{N,N-1}}{T_{N,N-1} + T_{N-1,N}} = \left[1 + \frac{T_{N-1,N}}{T_{N,N-1}}\right]^{-1}$$ , by PASTA,

where

$$T_{N,N-1} = \frac{1}{\mu_N \pi_+(0)} ,$$

$$T_{N-1,N} = \frac{1}{\lambda_{N-1}\pi_-(N-1)} = \frac{1}{\lambda\, E_{1,N-1}} .$$

# M/M/N+M (Erlang-A): Continued

Hence,

$$P(Wait > 0) = \left[1 + \frac{T_{N-1,N}}{T_{N,N-1}}\right]^{-1} = \left[1 + \frac{\pi_+(0)}{\rho\, E_{1,N-1}}\right]^{-1},$$

in which

$$\boldsymbol{E_{1,N-1}} = \frac{R^{N-1}/(N-1)!}{\sum_{k=0}^{N-1} R^k/k!} \quad ,$$

$$\pi_+(0) = \frac{(\lambda/\theta)^{N\mu/\theta}/(N\mu/\theta)!}{\sum_{i=0}^{\infty} (\lambda/\theta)^{N\mu/\theta+i}/(N\mu/\theta+i)!} \quad .$$

Recall: $E_{1,N-1} = P(Y = N-1 | Y \leq N-1)$,

where $Y \stackrel{d}{=} Pois(R)$, $\qquad R = \lambda/\mu$.

"Similarly"

$\pi_+(0) \stackrel{d}{=} P(X = N\mu/\theta | X \geq N\mu/\theta)$,

where $X \stackrel{d}{=} Pois(\lambda/\theta)$

Hence

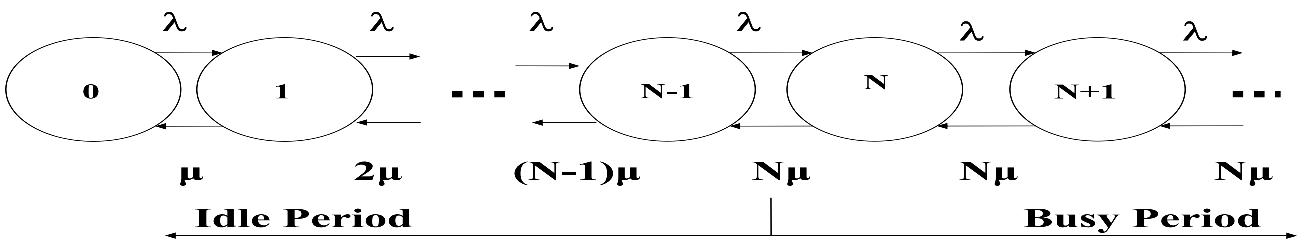
$$P(Wait > 0) = \left[1 + \frac{P(X = N\mu/\theta | X \geq N\mu/\theta)}{\rho\, P(Y = N-1 | Y \leq N-1)}\right]^{-1},$$

Note: Only $\pi_+(0)$ changes by the model.

# Special Cases: Erlang-C, Erlang-B, $\mu = \theta$

Erlang-C

Infinitely patient customers ($\theta = 0$)




$$P(Wait > 0) = E_{2,N} = \lim_{\theta \to 0} \left[ 1 + \frac{P(X = N\mu/\theta | X \geq N\mu/\theta)}{\rho \, P(Y = N-1 | Y \leq N-1)} \right]^{-1}$$

Lemma: Let $X \stackrel{d}{=} Pois(\lambda/\theta)$. Then

$$\lim_{\theta \to 0} P(X = N\mu/\theta | X \geq N\mu/\theta) = 1 - \rho.$$

Proof:

$$\lim_{\theta \to 0} \frac{(\lambda/\theta)^{N\mu/\theta}/(N\mu/\theta)!}{\sum_{i=0}^{\infty} (\lambda/\theta)^{N\mu/\theta+i}/(N\mu/\theta+i)!} =$$

$$\lim_{\theta \to 0} \left[ \sum_{i=0}^{\infty} (\tfrac{\lambda}{\theta})^i \cdot (\tfrac{\theta}{N\mu})^i / \prod_{j=1}^{i} (1 + \tfrac{j\theta}{N\mu}) \right]^{-1} =$$

$$\left[ \sum_{i=0}^{\infty} (\lambda/N\mu)^i \right]^{-1} = 1 - \rho.$$

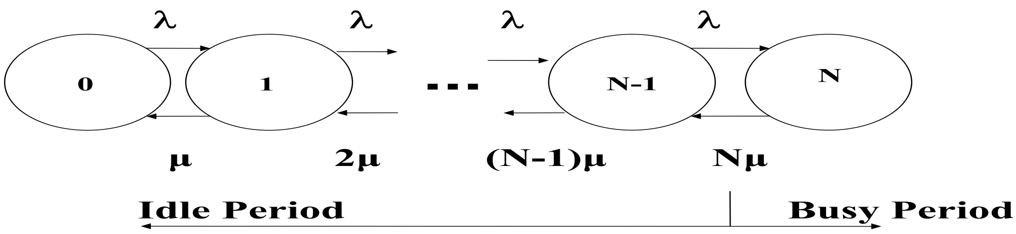
Conclude: Erlang-C Formula

$$E_{2,N} = \left[ 1 + \frac{1-\rho}{\rho \, E_{1,N-1}} \right]^{-1} .$$

# Special Cases, Continued

Erlang-B

Infinitely impatient customers ($\theta = \infty$)




$$P(Q_t \geq N) = \lim_{\theta \to \infty} \left[ 1 + \frac{\pi_+(0)}{\rho\, E_{1,N-1}} \right]^{-1}$$

Lemma: Let

$$\pi_+(0) = \frac{(\lambda/\theta)^{N\mu/\theta} \exp(-\lambda/\theta)}{\frac{N\mu}{\theta} \gamma(\frac{N\mu}{\theta}, \frac{\lambda}{\theta})}.$$

Then

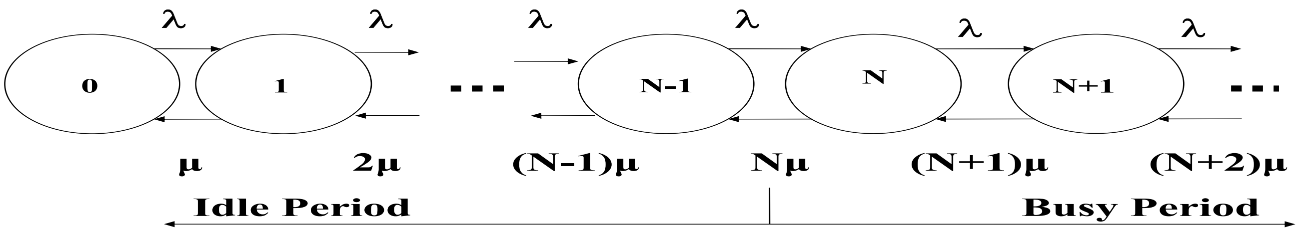
$$\lim_{\theta \to \infty} \pi_+(0) = 1.$$

Conclude: Erlang-B Formula

$$E_{1,N} = \left[ 1 + \frac{1}{\rho\, E_{1,N-1}} \right]^{-1}.$$

# Special Cases, Continued

$\mu = \theta \qquad (M/M/\infty)$




$$
\begin{aligned}
P(Wait > 0) &= \left[1 + \frac{P(Y = N | Y \geq N)}{\rho \, P(Y = N-1 | Y \leq N-1)}\right]^{-1}, \\
&\approx \left[1 + \frac{P(Y = N | Y \geq N)}{P(Y = N | Y \leq N)}\right]^{-1},
\end{aligned}
$$

for large $N$ and $\rho \approx 1$.

Recall: $\quad Y \stackrel{d}{=} Pois(R = \lambda/\mu)$.

## M/M/N/N (Erlang-B) with Many Servers: $N \uparrow \infty$

Assume: $\mu$ fixed, while $\lambda_N \uparrow \infty$ as $N \uparrow \infty$.

Recall: $R = R_N = \lambda_N/\mu$, $\rho = \rho_N = R_N/N$.

Erlang-B: $E_{1,N} = Pr\{Y_R = N\}/Pr\{Y_R \le N\}$,

where $Y_R \overset{d}{=}$ Poisson $(R)$.

$R$ large $Y_R \overset{d}{\approx} R + Z\sqrt{R}$, where $Z \overset{d}{=} N(0,1)$,

suggesting $N \sim R + \beta\sqrt{R}$, $-\infty < \beta < \infty$.

Then
$$
\begin{aligned}
E_{1,N} &= Pr\{N-1 < Y_R \le N\}/Pr\{Y_R \le N\} \\
&\approx Pr\left\{\beta - \tfrac{1}{\sqrt{R}} < Z \le \beta\right\}/Pr\{Z \le \beta\} \\
&\approx \tfrac{1}{\sqrt{R}}\,\phi(\beta)/\Phi(\beta) \approx \tfrac{1}{\sqrt{N}}\,\tfrac{\phi(\beta)}{\Phi(\beta)} = \\
&= \tfrac{1}{\sqrt{N}}\,\tfrac{\phi(-\beta)}{1-\Phi(-\beta)} = \tfrac{1}{\sqrt{N}}\,h(-\beta),
\end{aligned}
$$
$h =$ hazard rate of $N(0,1)$.

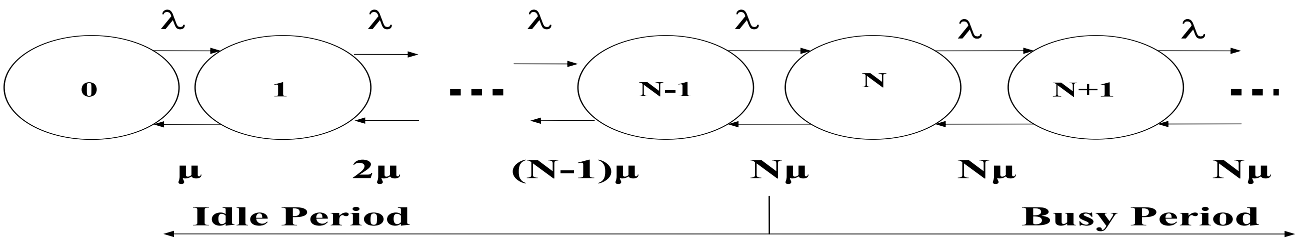
Algebra: $\mu$ fixed, $N \uparrow \infty$.

**QED:** $N \sim R + \beta\sqrt{R}$, for some $\beta$, $-\infty < \beta < \infty$

$$
\begin{aligned}
\Leftrightarrow\ \lambda &\sim \mu N - \beta\mu\sqrt{N} \\
\Leftrightarrow\ \rho &\sim 1 - \frac{\beta}{\sqrt{N}}, \quad \text{namely} \quad \lim_{N\to\infty} \sqrt{N}\,(1-\rho_N) = \beta.
\end{aligned}
$$

Theorem: (Jagerman, 1974) QED $\Leftrightarrow \lim_{N\to\infty} \sqrt{N}\,E_{1,N} = h(-\beta)$.

# M/M/N (Erlang-C) with Many Servers: $N \uparrow \infty$




$Q(0) = N$: all servers busy, no queue.

Recall $$E_{2,N} = \left[1 + \frac{T_{N-1,N}}{T_{N,N-1}}\right]^{-1} = \left[1 + \frac{1-\rho_N}{\rho_N E_{1,N-1}}\right]^{-1}.$$

Here $$T_{N-1,N} = \frac{1}{\lambda_N E_{1,N-1}} \sim \frac{1}{N\mu \times h(-\beta)/\sqrt{N}} \sim \frac{1/\mu}{h(-\beta)\sqrt{N}}$$

which applies as $\sqrt{N}\,(1-\rho_N) \to \beta,\ -\infty < \beta < \infty$.

Also $$T_{N,N-1} = \frac{1}{N\mu(1-\rho_N)} \sim \frac{1/\mu}{\beta\sqrt{N}}$$

which applies as above, but for $0 < \beta < \infty$.

Hence, $$E_{2,N} \sim \left[1 + \frac{\beta}{h(-\beta)}\right]^{-1},$$ assuming $\beta > 0$.

**QED:** 
$$\begin{aligned} & N \sim R + \beta\sqrt{R} \quad \text{for some } \beta, \quad 0 < \beta < \infty \\ \Leftrightarrow\ & \lambda \sim \mu N - \beta\mu\sqrt{N} \\ \Leftrightarrow\ & \rho \sim 1 - \frac{\beta}{\sqrt{N}}, \quad \text{namely} \quad \lim_{N\to\infty} \sqrt{N}\,(1-\rho_N) = \beta. \end{aligned}$$

Theorem (Halfin-Whitt, 1981) QED $\Leftrightarrow \lim_{N\to\infty} E_{2,N} = \left[1 + \frac{\beta}{h(-\beta)}\right]^{-1}$.

# QED Theorem (Halfin-Whitt, 1981)

Consider a sequence of $M/M/N$ models, $N = 1, 2, 3, \ldots$

Then the following **3 points of view** are equivalent:

- Customer $\lim_{N\to\infty} P_N(Wait > 0) = \alpha, \qquad 0 < \alpha < 1;$
- Server $\lim_{N\to\infty} \sqrt{N}(1 - \rho_N) = \beta, \qquad 0 < \beta < \infty;$
- Manager $N \approx R + \beta\sqrt{R}, \qquad R = \lambda \times E(S)$ large;

Here

$$\alpha = \left[1 + \frac{\beta\,\Phi(\beta)}{\phi(\beta)}\right]^{-1},$$

where $\phi(\cdot)/\Phi(\cdot)$ is the standard normal density/distribution.

Extremes:

**Everyone waits:** $\alpha = 1 \Leftrightarrow \beta = 0$ **Efficiency-driven**

**No one waits:** $\alpha = 0 \Leftrightarrow \beta = \infty$ **Quality-Driven**

## M/M/N+M (Erlang-A) with Many Servers: $\pi_+(0)$

Assume: $\mu,\ \theta$ fixed, while $\lambda_N \uparrow \infty$ as $N \uparrow \infty$.

Recall: $R = R_N = \lambda_N/\mu, \qquad \rho = \rho_N = R_N/N.$

$$\pi_+(0) = Pr\{X_\lambda = N\mu/\theta\}/Pr\{X_\lambda \geq N\mu/\theta\},$$

where $X_\lambda \stackrel{d}{=}$ Poisson $(\lambda/\theta)$.

$N$ large $\quad X_\lambda \stackrel{d}{\approx} \lambda/\theta + Z\sqrt{\lambda/\theta}$, where $Z \stackrel{d}{=} N(0,1)$,

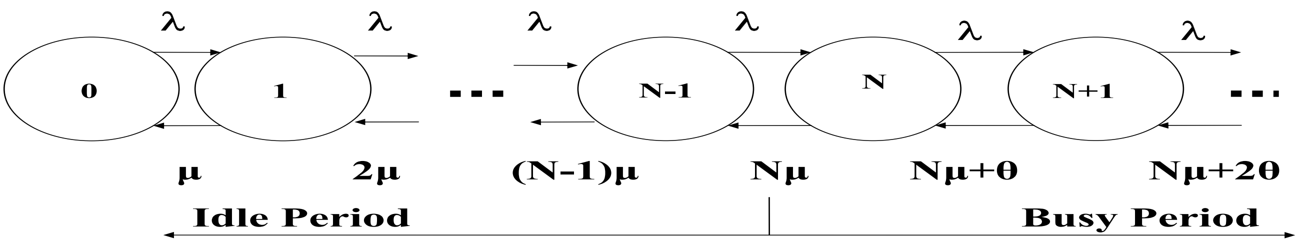
suggesting $\quad N \sim R + \beta\sqrt{R},\ -\infty < \beta < \infty.$

Then
$$\begin{aligned}
\pi_+(0) &= Pr\{N\mu/\theta - 1 < X_\lambda \leq N\mu/\theta\}/Pr\{X_\lambda \geq N\mu/\theta\} \\
&\approx Pr\left\{\delta - \frac{1}{\sqrt{\lambda/\theta}} < Z \leq \delta\right\}/Pr\{Z \geq \delta\},\ \delta = \beta\sqrt{\mu/\theta} \\
&\approx \frac{1}{\sqrt{\lambda/\theta}}\,\phi(\delta)/(1-\Phi(\delta)) \approx \frac{1}{\sqrt{N}}\sqrt{\frac{\theta}{\mu}}\,\frac{\phi(\delta)}{1-\Phi(\delta)} = \\
&= \frac{1}{\sqrt{N}}\sqrt{\frac{\theta}{\mu}}\,h(\delta)\ ,\ h = \text{hazard rate of } N(0,1).
\end{aligned}$$

Algebra: $\mu, \theta$ fixed, $N \uparrow \infty$.

Proposition: QED $\Leftrightarrow \lim_{N\to\infty} \sqrt{N}\,\pi_+(0) = \sqrt{\frac{\theta}{\mu}}h(\delta)$.

# M/M/N+M (Erlang-A) with Many Servers: QED




$Q(0) = N$: all servers busy, no queue.

Recall $$P(Wait > 0) = \left[1 + \frac{T_{N-1,N}}{T_{N,N-1}}\right]^{-1} = \left[1 + \frac{\pi_+(0)}{\rho_N E_{1,N-1}}\right]^{-1}.$$

Here $$T_{N-1,N} = \frac{1}{\lambda_N E_{1,N-1}} \sim \frac{1}{N\mu \times h(-\beta)/\sqrt{N}} \sim \frac{1}{\sqrt{N}} \frac{1/\mu}{h(-\beta)}$$

Also $$T_{N,N-1} = \frac{1}{N\mu\pi_+(0)} \sim \frac{1}{\sqrt{N}} \frac{1/\mu}{h(\delta)/\delta}, \qquad \delta = \beta\sqrt{\mu/\theta}$$

Both apply as $\sqrt{N}(1-\rho_N) \to \beta, \ -\infty < \beta < \infty$.

Hence, $$P(Wait > 0) \sim \left[1 + \frac{h(\delta)/\delta}{h(-\beta)/\beta}\right]^{-1},$$ Garnett Function.

**QED:**
$$N \sim R + \beta\sqrt{R} \quad \text{for some } \beta, \quad -\infty < \beta < \infty$$
$$\Leftrightarrow \quad \lambda \sim \mu N - \beta\mu\sqrt{N}$$
$$\Leftrightarrow \quad \rho \sim 1 - \frac{\beta}{\sqrt{N}}, \quad \text{namely} \quad \lim_{N\to\infty} \sqrt{N}(1-\rho_N) = \beta.$$

Theorem (Garnett, M., Reiman 2002)

QED $\Leftrightarrow \lim_{N\to\infty} P(Wait > 0) = \left[1 + \frac{h(\delta)/\delta}{h(-\beta)/\beta}\right]^{-1}.$

## QED Theorem (Garnett, M., Reiman '02; Zeltyn '03)

Consider a sequence of $M/M/N+G$ models, $N = 1, 2, 3, \ldots$

Then the following **points of view** are equivalent:

- **QED** $\quad \%\{Wait > 0\} \approx \alpha, \qquad 0 < \alpha < 1;$
- Customer $\quad \%\{Abandon\} \approx \frac{\gamma}{\sqrt{N}}, \qquad 0 < \gamma;$
- Server $\quad OCC \approx 1 - \frac{\beta + \gamma}{\sqrt{N}}, \qquad -\infty < \beta < \infty;$
- Manager $\quad N \approx R + \beta\sqrt{R}, \qquad R = \lambda \times E(S)$ not small;

QED performance (ASA,...) is easily computable, all in terms of $\beta$

(the square-root staffing level)

Covers also the Extremes:

$\alpha = 1:$ $\quad \mathbf{N = R - \gamma R}$ $\quad$ **Efficiency-driven**

$\alpha = 0:$ $\quad N = R + \gamma R$ $\quad$ **Quality-Driven**