Queues in Hospitals: Semi-Open Queueing Networks in the QED Regime

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Agenda

- Introduction
- Medical Unit Model
- Mathematical Results
- Numerical Example
- Time-varying Model
- Future Research

Work-Force and Bed Capacity Planning

- Total health expenditure as percentage of gross domestic product: Israel 8%, EU 10%, USA 14%.
- Human resource constitute 70% of hospital expenditure.
- There are 3M registered nurses in the U.S. but still a chronic shortage.
- California law set nurse-to-patient ratios such as 1:6 for pediatric care unit.
- O.B. Jennings and F. de Véricourt (2008) showed that fixed ratios do not account for economies of scale.
- Management measures average occupancy levels, while arrivals have seasonal patterns and stochastic variability (Green 2004).

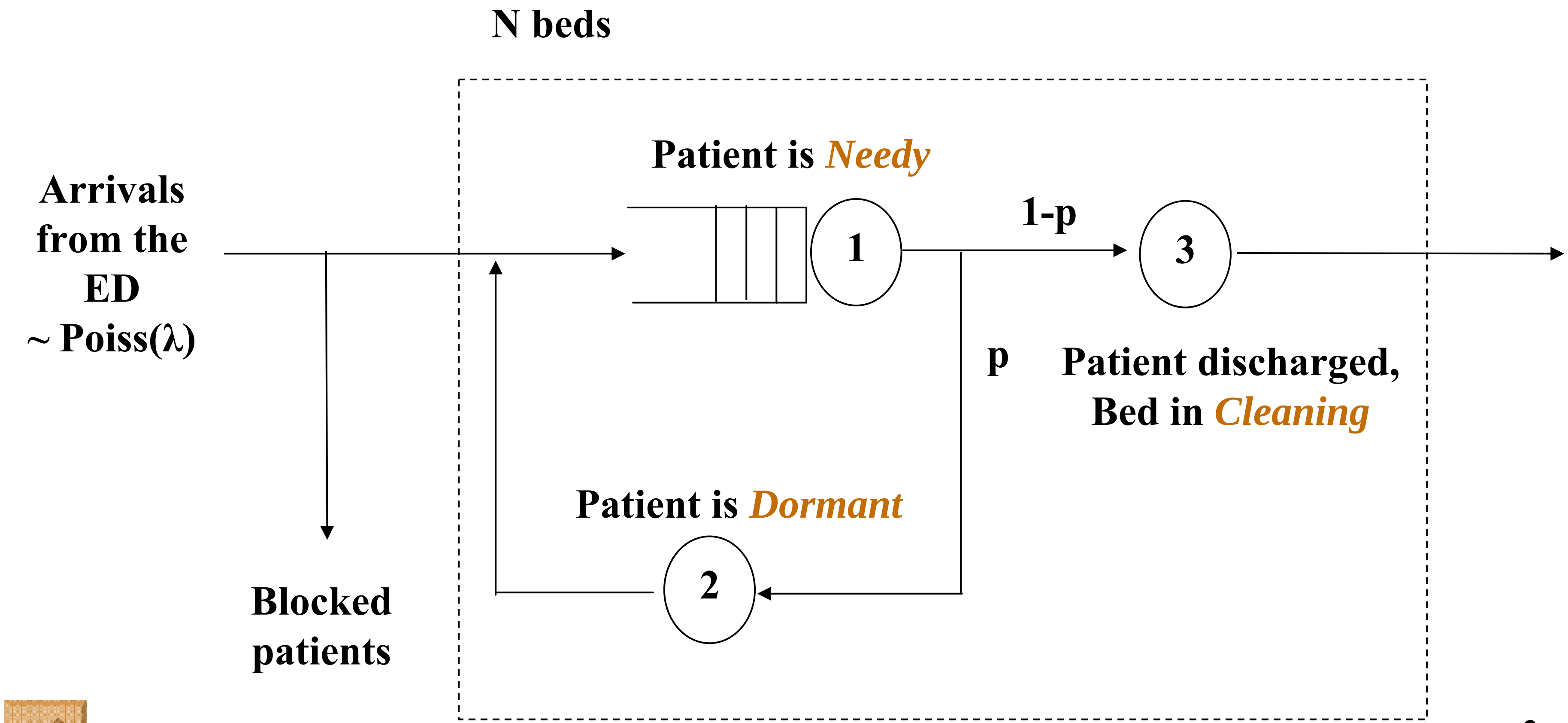
Research Objectives

- Analyzing model for a Medical Unit with *s* nurses and *n* beds, which are partly/fully occupied by patients: semi-open queueing network with multiple statistically identical customers and servers.
- Questions addressed: How many servers (nurses) are required (staffing), and how many fixed resources (beds) are needed (allocation) in order to minimize costs while sustaining a certain service level?
- Coping with time-variability

We Follow -

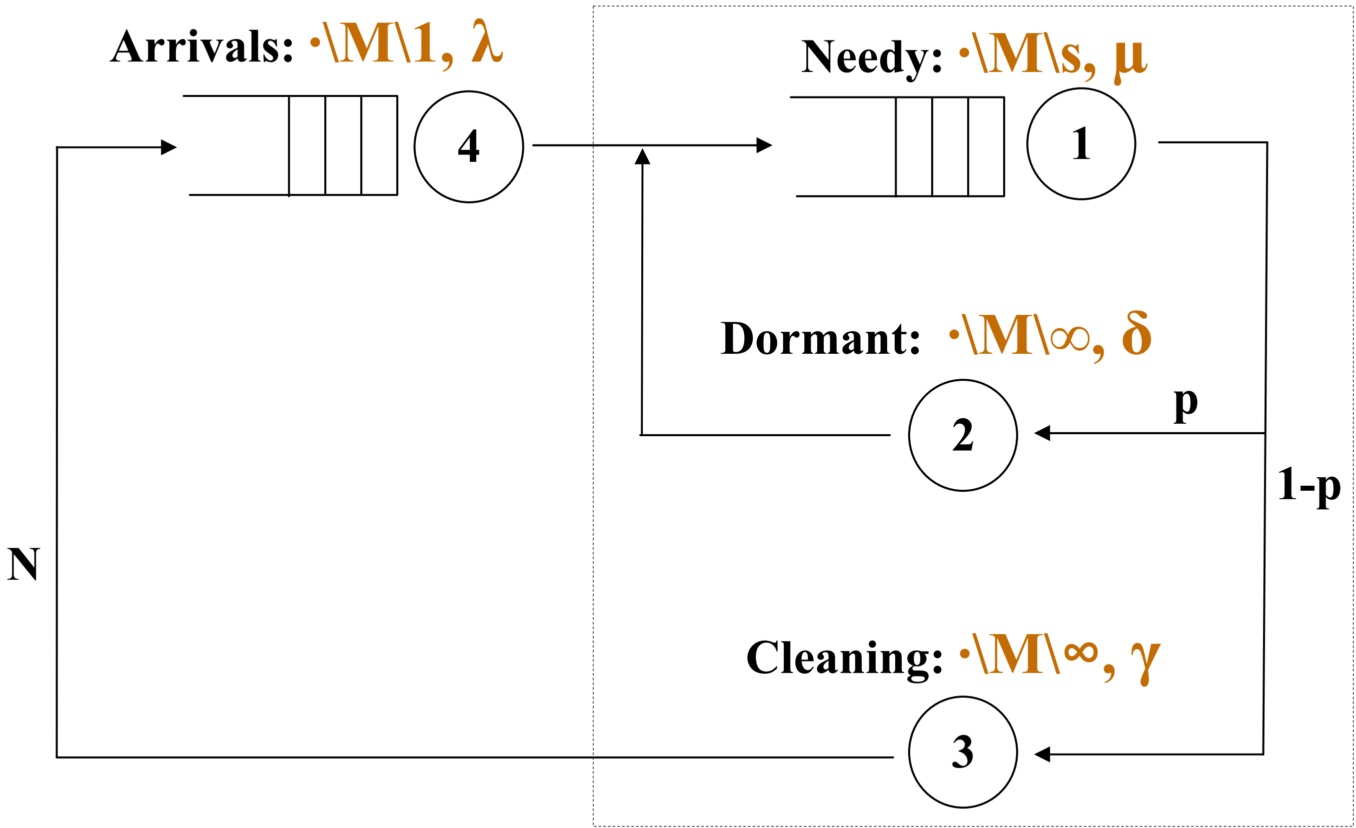
- Basic:
  - Halfin and Whitt (1981)
  - Mandelbaum, Massey and Reiman (1998)
  - Khudyakov (2006)
- Analytical models in HC:
  - Nurse staffing: Jennings and Véricourt (2007), Yankovic and Green (2007)
  - Beds capacity: Green (2002,2004)
- Service Engineering (mainly call centers):
  - Gans, Koole, Mandelbaum: "Telephone call centers: Tutorial, Review and Research prospects"

The MU Model as a Semi-Open Queueing Network



Service times are Exponential; Routing is Markovian

The MU Model as a Closed Jackson Network



→Product Form - π_N(n,d,c) stationary dist.

Stationary Distribution

$$\pi(i,j,k) = \begin{cases} \pi_0 \frac{1}{\nu(i)} \left(\frac{\lambda}{(1-p)\mu}\right)^i \frac{1}{j!} \left(\frac{p\lambda}{(1-p)\delta}\right)^j \frac{1}{k!} \left(\frac{\lambda}{\gamma}\right)^k & , \quad 0 \le i+j+k \le n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Here $\nu(i)$ is defined as

$$\nu(i) := \begin{cases} i! & , \quad i \le s, \\ s! s^{i-s} & , \quad i \ge s, \end{cases}$$

where π_0 is given by (see Appendix A)

$$\pi_0^{-1} = \sum_{0 \le i+j+k \le n} \frac{1}{\nu(i)} \left(\frac{\lambda}{(1-p)\mu}\right)^i \frac{1}{j!} \left(\frac{p\lambda}{(1-p)\delta}\right)^j \frac{1}{k!} \left(\frac{\lambda}{\gamma}\right)^k$$

$$= \sum_{l=0}^{n} \frac{1}{l!} \left(\frac{\lambda}{(1-p)\mu} + \frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma}\right)^l$$

$$+ \sum_{l=s+1}^{n} \sum_{m=s+1}^{l} \sum_{i=s+1}^{m} \left(\frac{1}{s! s^{i-s}} - \frac{1}{i!}\right) \frac{1}{(m-i)!(l-m)!} \left(\frac{\lambda}{(1-p)\mu}\right)^i \cdot$$

$$\left(\frac{p\lambda}{(1-p)\delta}\right)^{m-i} \left(\frac{\lambda}{\gamma}\right)^{l-m}. \quad (2.1)$$

Service Level Objectives

(Function of $\lambda,\mu,\delta,\gamma,p,s,n$)

- Blocking probability
- Delay probability
- Probability of timely service (wait more than *t*)
- Expected waiting time
- Average occupancy level of beds
- Average utilization level of nurses

Blocking probability

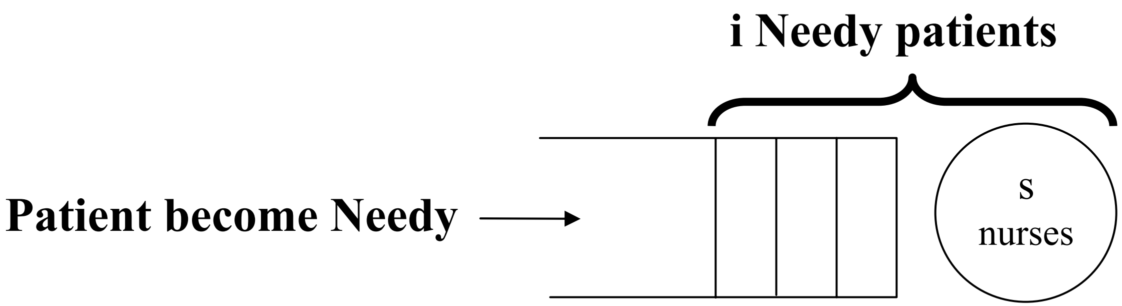
- The probability to have l occupied beds in the ward:

$$P_l := \sum_{\substack{i,j,k \geq 0 \\ i+j+k=l}} \pi(i,j,k) = \sum_{i=0}^{l} \sum_{j=0}^{l-i} \pi(i,j,l-i-j)$$

$$P_l = \pi_0 \left(\frac{1}{l!} \left(\frac{\lambda}{(1-p)\mu} + \frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma} \right)^l \right.$$
$$\left. + I_{\{l>s\}} \sum_{i=s+1}^{l} \sum_{j=0}^{l-i} \left(\frac{1}{s! s^{i-s}} - \frac{1}{i!} \right) \left(\frac{\lambda}{(1-p)\mu} \right)^i \frac{1}{j!} \left(\frac{p\lambda}{(1-p)\delta} \right)^j \frac{1}{(l-i-j)!} \left(\frac{\lambda}{\gamma} \right)^{l-i-j} \right)$$

Probability of timely service and the Delay Probability

- What happens when a patient becomes needy ?
  - He will need to wait an in-queue random waiting time that follows an Erlang distribution with $(i-s+1)^+$ stages, each with rate μs.
- What is the probability that this patient will find i other needy patients?
  - We need to use the Arrival Theorem


i Needy patients
Patient become Needy
s
nurses

The Arrival Theorem

The Arrival Theorem. *In a closed Jackson network, the arrival at (or the departure from) any node observes time averages, with the job itself excluded. In particular, the probability that the network is in state*[2] $x - e_i$ *immediately before an arrival (or immediately after a departure) epoch at node* i *is equal to the ergodic distribution, of a closed network with one fewer job, in state* $x - e_i$.

Thus, the probability that a patient that become needy will see i needy patients in the system is $\pi_{n-1}(i,j,k)$

Probability of timely service and the Delay Probability

$$P(W=0)=\sum_{l=0}^{n-1}\sum_{m=0}^{l}\sum_{i=0}^{min\{m,s-1\}}\pi^A(i,m-i,l-m)$$

$$P(W\leq t)=P(W=0)+\sum_{i=s}^{n-1}P(\text{there are } (i-s+1) \text{ patients who ended}$$

$$\text{their service on time} \leq t|\text{Arrival at the needy state found } i \text{ needy patients})\cdot$$

$$\cdot\,\pi^A(i,m-i,l-m)=$$

$$=1-\sum_{l=s}^{n-1}\sum_{m=s}^{l}\sum_{i=s}^{m}\pi_{n-1}(i,m-i,l-m)\sum_{h=0}^{i-s}\frac{(\mu st)^h}{h!}e^{-\mu st}.$$

$$P(W>t)=\sum_{l=s}^{n-1}\sum_{m=s}^{l}\sum_{i=s}^{m}\pi_{n-1}(i,m-i,l-m)\sum_{h=0}^{i-s}\frac{(\mu st)^h}{h!}e^{-\mu st}$$

$$P(W>0)=\sum_{i\geq s}\pi_{n-1}(i,j,k)=\sum_{l=s}^{n-1}\sum_{m=s}^{l}\sum_{i=s}^{m}\pi_{n-1}(i,m-i,l-m)$$

Expected waiting time

- via the tail formula:

$$
\begin{aligned}
E[W] &= \int_0^\infty P(W > t)dt = \int_0^\infty \sum_{l=s}^{n-1} \sum_{m=s}^{l} \sum_{i=s}^{m} \pi_{n-1}(i, m-i, l-m) \sum_{h=0}^{i-s} \frac{(\mu s t)^h}{h!} e^{-\mu s t} dt \\
&= \sum_{l=s}^{n-1} \sum_{m=s}^{l} \sum_{i=s}^{m} \pi_{n-1}(i, m-i, l-m) \sum_{h=0}^{i-s} \int_0^\infty \frac{(\mu s t)^h}{h!} e^{-\mu s t} dt \\
&= \sum_{l=s}^{n-1} \sum_{m=s}^{l} \sum_{i=s}^{m} \pi_{n-1}(i, m-i, l-m) \sum_{h=0}^{i-s} \frac{1}{\mu s} \\
&= \frac{1}{\mu s} \sum_{l=s}^{n-1} \sum_{m=s}^{l} \sum_{i=s}^{m} \pi_{n-1}(i, m-i, l-m)(i-s+1).
\end{aligned}
$$

QED Q's: Quality- and Efficiency-Driven Queues

- Traditional queueing theory predicts that service-quality and server's efficiency **must** trade off against each other.
- Yet, one can balance both requirements carefully (Example: in well-run call-centers, 50% served "immediately", along with over 90% agent's utilization, is not uncommon)
- This is achieved in a special asymptotic operational regime – the QED regime

QED Regime characteristics

- High service quality
- High resource efficiency
- Square-root staffing rule

The offered load at service station 1 (needy)

The offered load at non-service station 2+3 (dormant + cleaning)

$$\text{(i)}\ s = \frac{\lambda}{(1-p)\mu} + \beta\sqrt{\frac{\lambda}{(1-p)\mu}} + o(\sqrt{\lambda}), \qquad -\infty < \beta < \infty$$

$$\text{(ii)}\ n - s = \eta\sqrt{\frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma}} + \frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma} + o(\sqrt{\lambda}), \quad -\infty < \eta < \infty$$

- Many-server asymptotic

Probability of Delay

$$P(W>0)=\sum_{i\geq s}\pi_{n-1}(i,j,k)=\sum_{l=s}^{n-1}\sum_{m=s}^{l}\sum_{i=s}^{m}\pi_{n-1}(i,m-i,l-m)$$

Theorem 2. *Let the variables λ, s and n tend to ∞ simultaneously and satisfy the QED conditions. Then*

$$\lim_{\lambda\to\infty}P(W>0)=\left(1+\frac{\int_{-\infty}^{\beta}\Phi\left(\eta+(\beta-t)\sqrt{B}\right)d\Phi(t)}{\frac{\phi(\beta)\Phi(\eta)}{\beta}-\frac{\phi(\sqrt{\eta^2+\beta^2})}{\beta}e^{\frac{1}{2}\eta_1^2}\Phi(\eta_1)}\right)^{-1}$$

where $B=\frac{R_N}{R_C+R_D}=\frac{\delta\gamma}{\mu(p\gamma+(1-p)\delta)}$, $\eta_1=\eta-\beta\sqrt{B^{-1}}$.

- The probability is a function of three parameters: beta, eta, and offered-load-ratio

Expected Waiting Time

$$E[W] = \frac{1}{\mu s}\sum_{l=s}^{n-1}\sum_{m=s}^{l}\sum_{i=s}^{m}\pi_{n-1}(i, m-i, l-m)(i-s+1)$$

Theorem 4. *Let the variables* λ, s *and* n *tend to* ∞ *simultaneously and satisfy the QED conditions. Then*

$$\lim_{\lambda\to\infty}\sqrt{s}E[W] = \frac{1}{\mu}\frac{\frac{\phi(\beta)\Phi(\eta)}{\beta}\frac{1}{\beta} + \frac{\phi(\sqrt{\eta^2+\beta^2})}{\beta}e^{\frac{1}{2}\eta_1^2}\Phi(\eta_1)\left(\frac{\beta}{B} - \frac{\eta}{\sqrt{B}} - \frac{1}{\beta}\right)}{\int_{-\infty}^{\beta}\Phi\left(\eta + (\beta - t)\sqrt{B}\right)d\Phi(t) + \frac{\phi(\beta)\Phi(\eta)}{\beta} - \frac{\phi(\sqrt{\eta^2+\beta^2})}{\beta}e^{\frac{1}{2}\eta_1^2}\Phi(\eta_1)}$$

where $B = \frac{R_N}{R_C+R_D} = \frac{\delta\gamma}{\mu(p\gamma+(1-p)\delta)}$, $\eta_1 = \eta - \beta\sqrt{B^{-1}}$.

- Waiting time is one order of magnitude less then the service time.

Probability of Blocking

$$P_l = \pi_0 \left(\frac{1}{l!}\left(\frac{\lambda}{(1-p)\mu} + \frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma}\right)^l \right.$$

$$\left. + I_{\{l>s\}} \sum_{i=s+1}^{l}\sum_{j=0}^{l-i}\left(\frac{1}{s!s^{i-s}} - \frac{1}{i!}\right)\left(\frac{\lambda}{(1-p)\mu}\right)^i \frac{1}{j!}\left(\frac{p\lambda}{(1-p)\delta}\right)^j \frac{1}{(l-i-j)!}\left(\frac{\lambda}{\gamma}\right)^{l-i-j}\right)$$

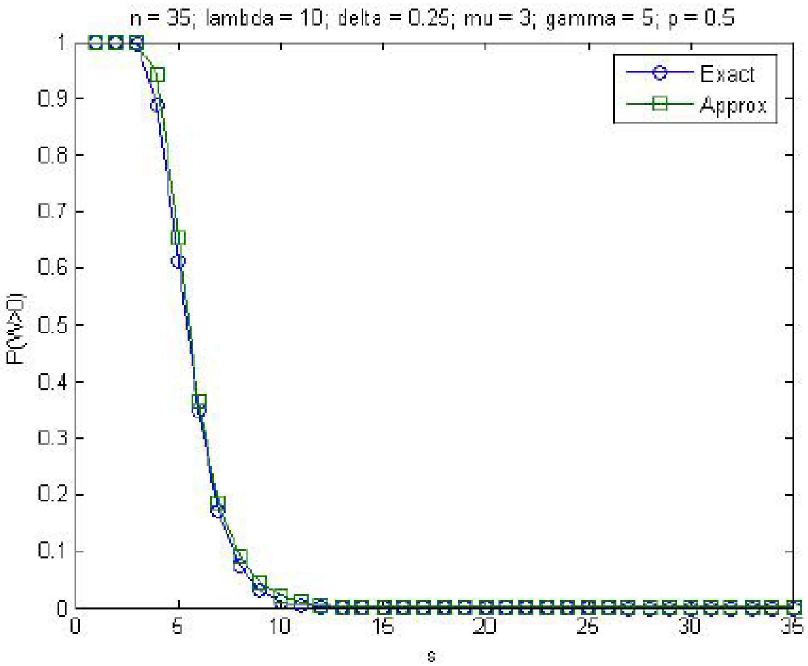
Theorem 6. *Let the variables λ, s and n tend to ∞ simultaneously and satisfy the QED conditions. Define $B = \frac{R_N}{R_C+R_D} = \frac{\delta\gamma}{\mu(p\gamma+(1-p)\delta)}$, then*

$$\lim_{\lambda\to\infty}\sqrt{s}P(block) = \frac{\nu\phi(\nu_1)\Phi(\nu_2)+\phi(\sqrt{\eta^2+\beta^2})e^{\frac{\eta_1^2}{2}}\Phi(\eta_1)}{\int_{-\infty}^{\beta}\Phi\left(\eta+(\beta-t)\sqrt{B}\right)d\Phi(t) + \frac{\phi(\beta)\Phi(\eta)}{\beta} - \frac{\phi(\sqrt{\eta^2+\beta^2})}{\beta}e^{\frac{1}{2}\eta_1^2}\Phi(\eta_1)} \qquad (5.9)$$

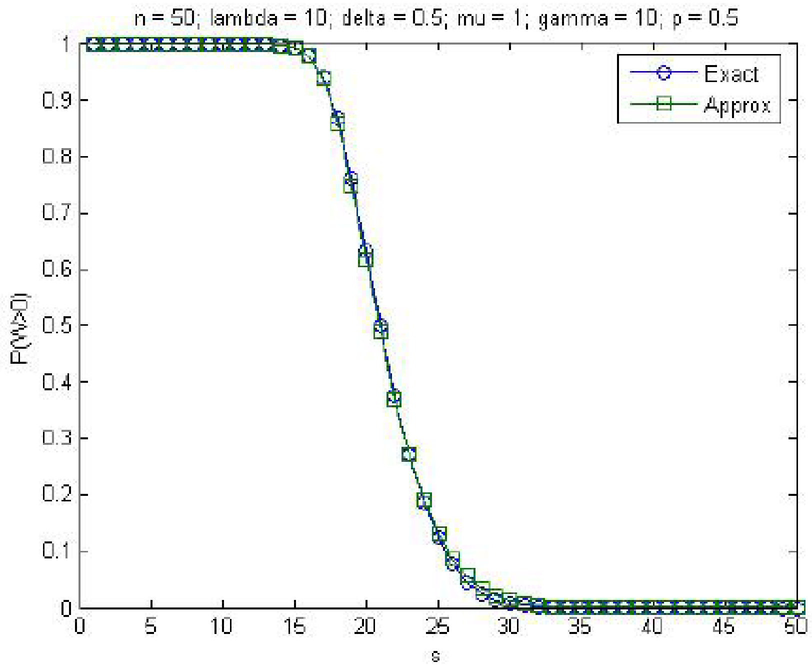
where $\eta_1 = \eta - \frac{\beta}{\sqrt{B}}$, $\nu = \frac{1}{\sqrt{1+B^{-1}}}$, $\nu_1 = \frac{\eta\sqrt{B^{-1}}+\beta}{\sqrt{1+B^{-1}}}$, $\nu_2 = \frac{\beta\sqrt{B^{-1}}-\eta}{\sqrt{1+B^{-1}}}$.

- P(Blocking) << P(Waiting)

Approximation vs. Exact Calculation – Medium system (n=35,50), P(W>0)

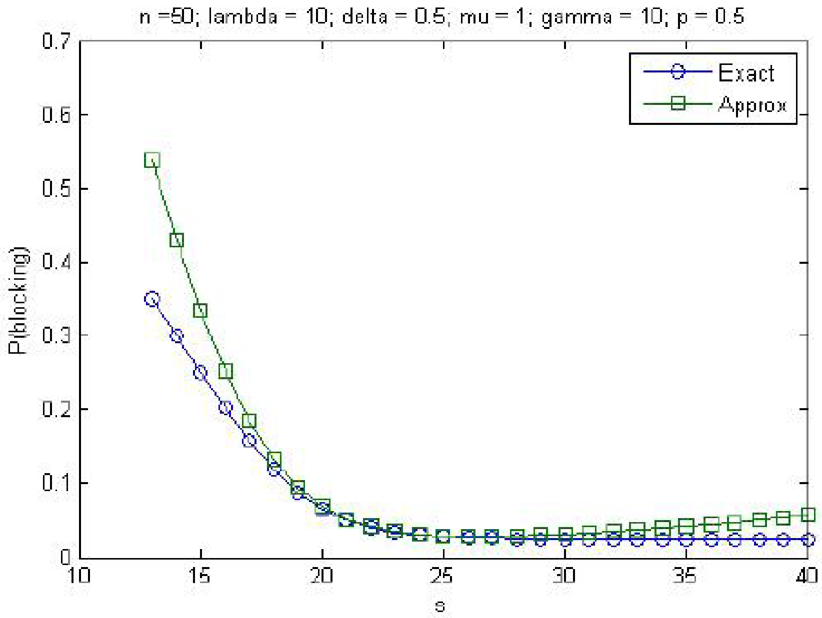


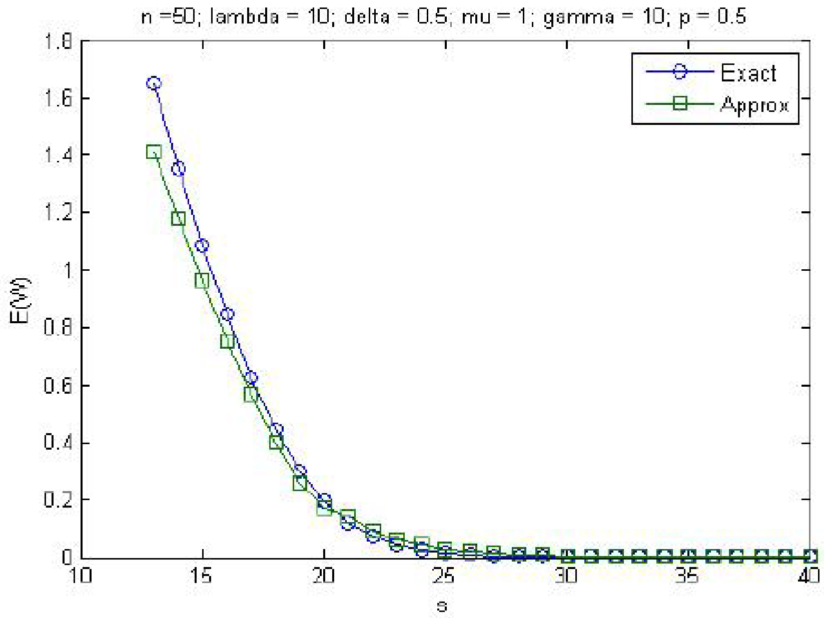
(a)



(b)

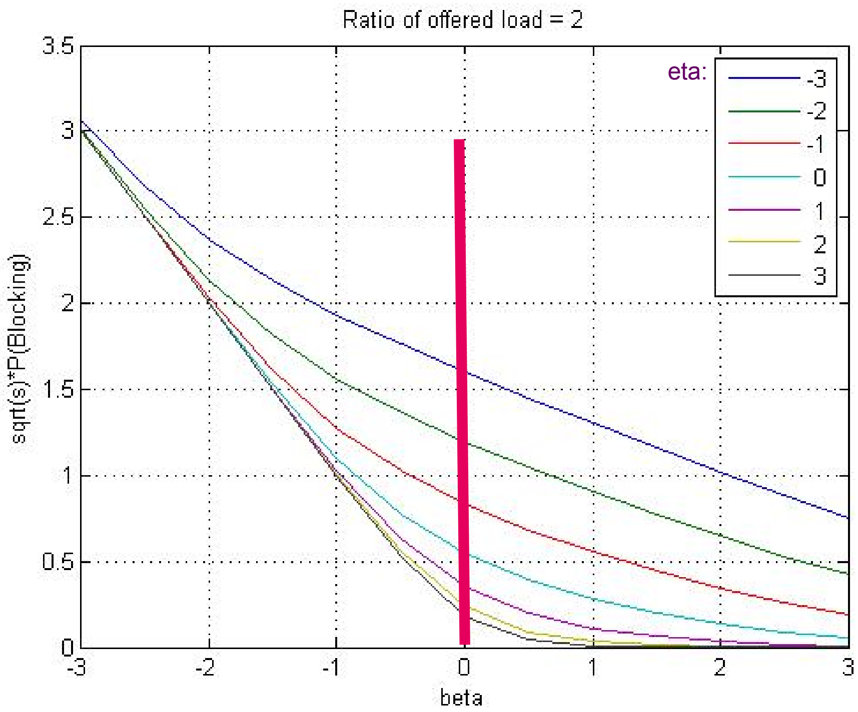
Approximation vs. Exact Calculation – P(blocking) and E[W]



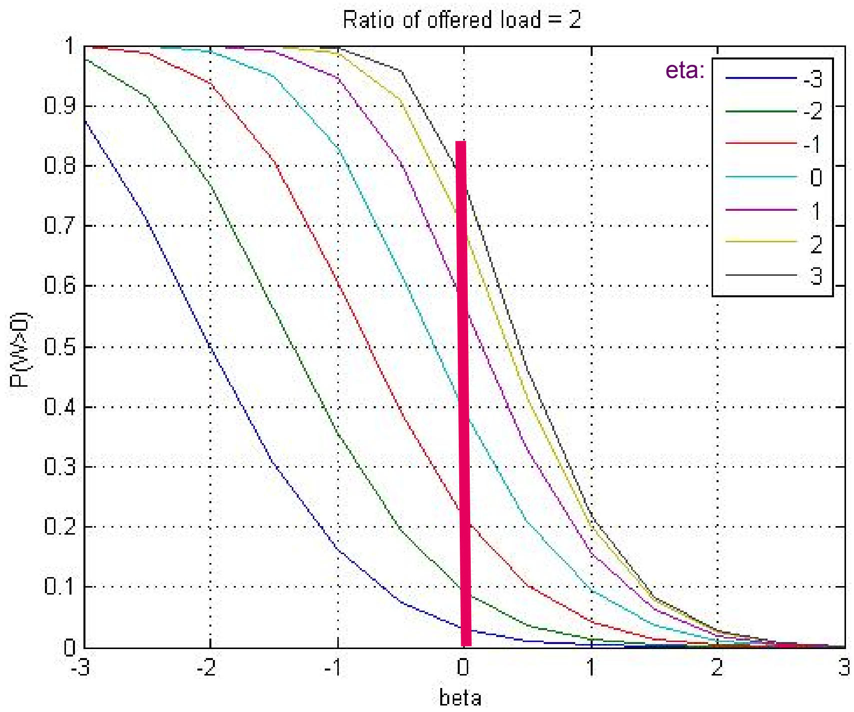


The influence of β and η?

Blocking

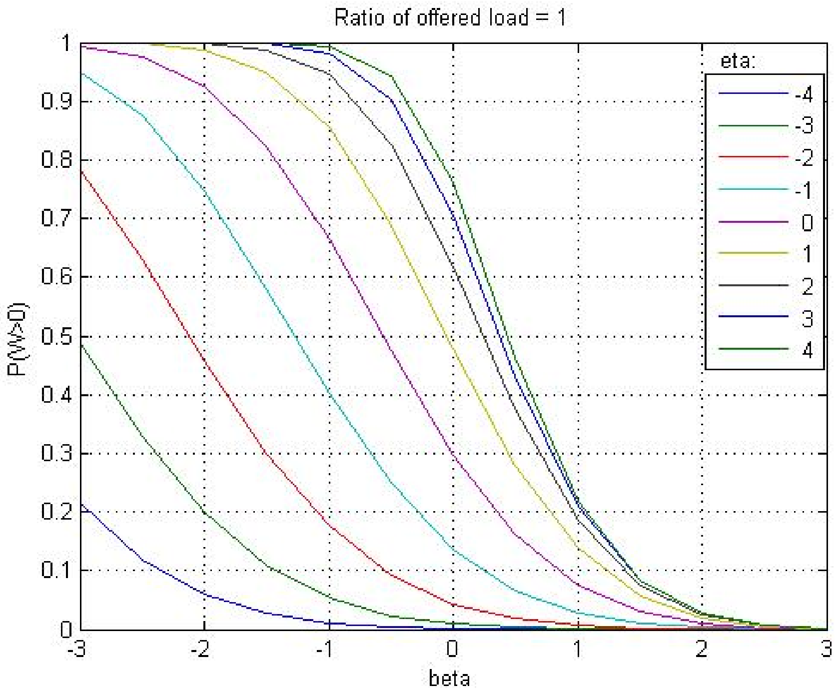


Waiting

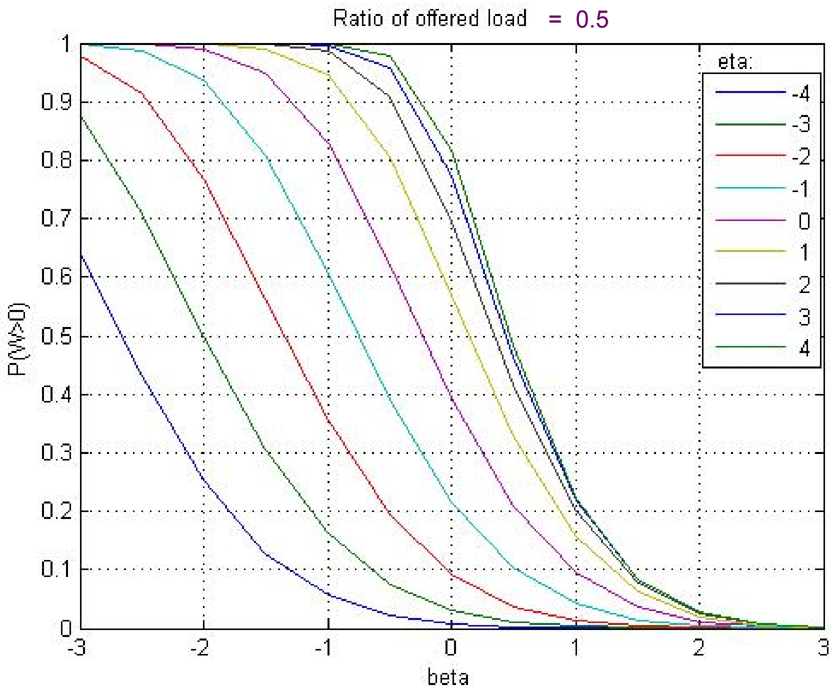


$$\text{(i)}\ s = \frac{\lambda}{(1-p)\mu} + \beta\sqrt{\frac{\lambda}{(1-p)\mu}} + o(\sqrt{\lambda}), \qquad -\infty < \beta < \infty$$

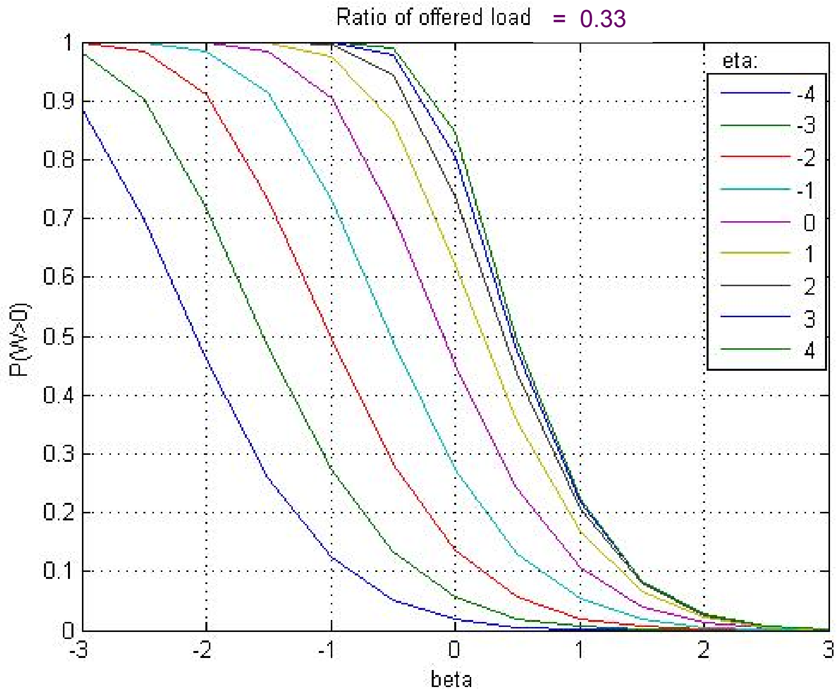
$$\text{(ii)}\ n - s = \eta\sqrt{\frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma}} + \frac{p\lambda}{(1-p)\delta} + \frac{\lambda}{\gamma} + o(\sqrt{\lambda}), \quad -\infty < \eta < \infty$$



Ratio of offered load = 1
P(W>0)
beta
eta:
-4
-3
-2
-1
0
1
2
3
4



Ratio of offered load = 0.5
P(W>0)
beta
eta:
-4
-3
-2
-1
0
1
2
3
4



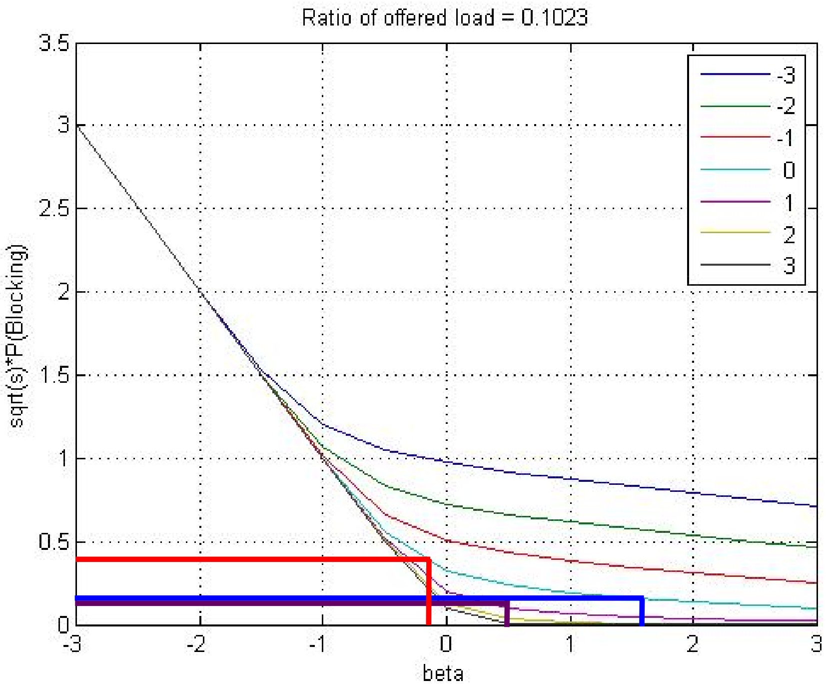
Ratio of offered load = 0.33
P(W>0)
beta
eta:
-4
-3
-2
-1
0
1
2
3
4

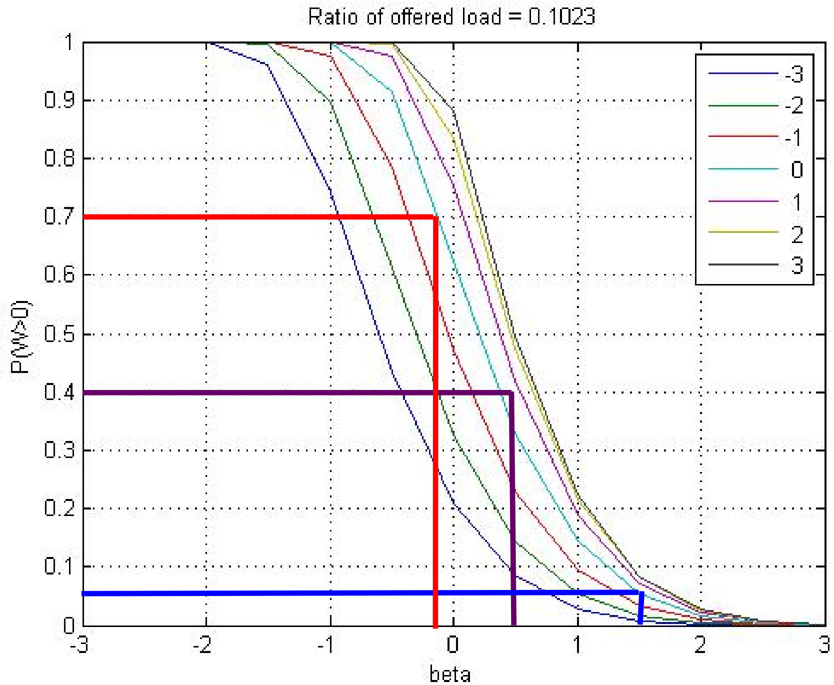
Numerical Example

(based on Lundgren and Segesten 2001 + Yankovic and Green 2007)

- N=42 with 78% occupancy
- ALOS = 4.3 days
- Average service time = 15 min
- 0.4 requests per hour
- => $\lambda = 0.32$, $\mu=4$, $\delta=0.4$, $\gamma=4$, $p=0.975$
- => Ratio of offered load = 0.1

How to find the required β and η?





if β=0.5 and η=0.5 (s=4, n=38): P(block)≅0.07, P(wait) ≅ 0.4

if β=1.5 and η ≅ 0 (s=6, n=37): P(block)≅0.068, P(wait) ≅ 0.084

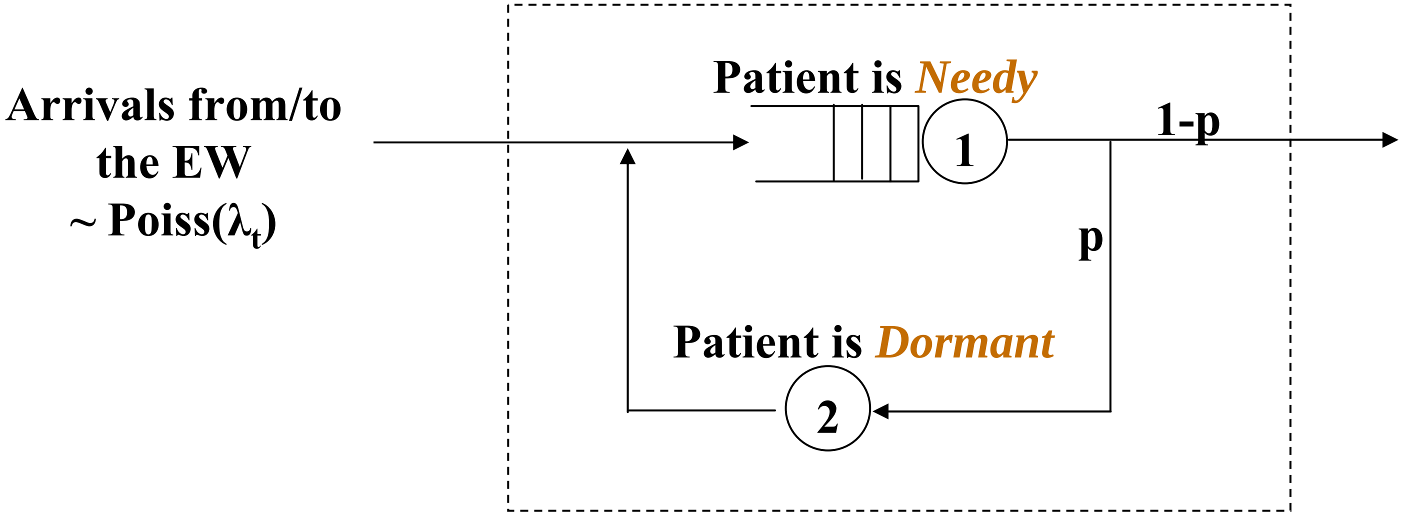
if β=-0.1 and η ≅ 0 (s=3, n=34): P(block)≅0.21, P(wait) ≅ 0.70

Regime	N	S	P(wait)	P(blocking)	E[W]
Nurses – QED; Beds-ED	7	4	0.32	0.84	0.11
Nurses – QD ; Beds-ED	7	7	0	0.83	0
Nurses – QED; Beds-QED	20	10	0.44	0.55	0.11
Nurses – QD ; Beds-QED	20	15	0	0.53	0
Nurses – ED ; Beds-ED	30	3	1	0.85	7.89
Nurses – QED; Beds-QED	30	14	0.55	0.35	0.13
Nurses – QD ; Beds-QED	30	21	0	0.31	0
Nurses – ED ; Beds-ED	50	3	1	0.85	14.56
Nurses – ED ; Beds-QED	50	10	1	0.5	2.85
Nurses – QED; Beds-QD	50	21	0.5	0.05	0.12
Nurses – QD ; Beds-QD	50	31	0	0.02	0

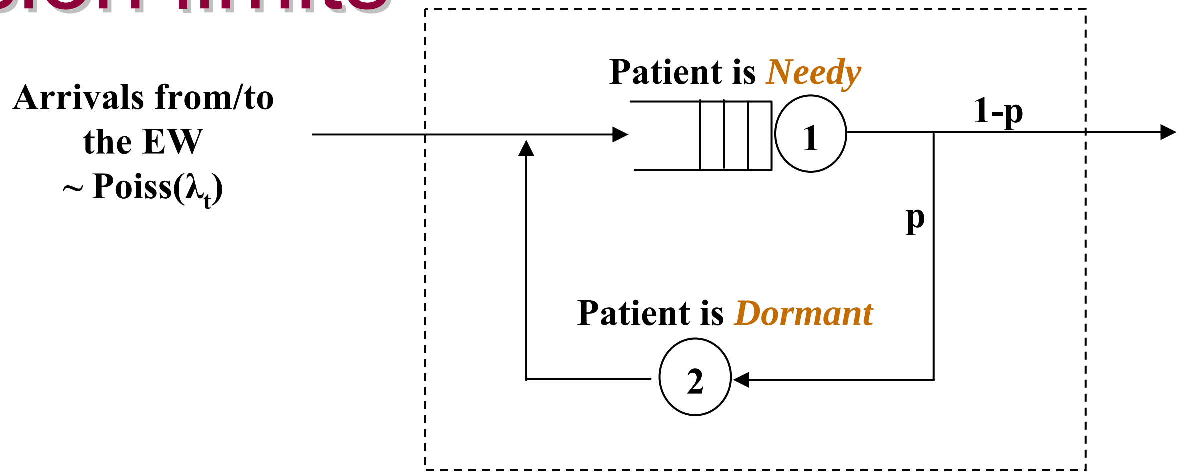
Lambda-10; Delta-0.5; Mu-1; Gamma-10; p-0.5; Ratio=1.05

Modeling time-variability

- Procedures at mass-casualty event
- Blocking cancelled -> open system



Fluid and Diffusion limits



$$Q_1(t) = Q_1(0) + A_1^a\left(\int_0^t \lambda_s ds\right) - A_2^d\left(\int_0^t p\mu_s\left(Q_1(s)\wedge n_s\right)ds\right) - A_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1(s)\wedge n_s\right)ds\right)$$
$$+ A_{21}\left(\int_0^t \delta_s Q_2(s)ds\right)$$
$$Q_2(t) = Q_2(0) + A_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1(s)\wedge n_s\right)ds\right) - A_{21}\left(\int_0^t \delta_s Q_2(s)ds\right),$$

where A_1^a, A_2^d, A_{12} and A_{21} are four mutually independent, standard (mean rate 1), Poisson processes.

Scaling

- The arrival rate and the number of nurses grow together to infinity, i.e. scaled up by η.

$$
\begin{aligned}
Q_1^\eta(t) &= Q_1^\eta(0) + A_1^a\left(\int_0^t \eta\lambda_s ds\right) - A_2^d\left(\int_0^t p\mu_s\left(Q_1^\eta(s)\wedge \eta n_s\right)ds\right)\\
&\quad - A_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1^\eta(s)\wedge \eta n_s\right)ds\right) + A_{21}\left(\int_0^t \delta_s Q_2^\eta(s)ds\right)\\
&= Q_1^\eta(0) + A_1^a\left(\int_0^t \eta\lambda_s ds\right) - A_2^d\left(\int_0^t \eta p\mu_s\left(\frac{1}{\eta}Q_1^\eta(s)\wedge n_s\right)ds\right)\\
&\quad - A_{12}\left(\int_0^t \eta(1-p)\mu_s\left(\frac{1}{\eta}Q_1^\eta(s)\wedge n_s\right)ds\right) + A_{21}\left(\int_0^t \eta\delta_s\left(\frac{1}{\eta}Q_2^\eta(s)\right)ds\right),\\
Q_2^\eta(t) &= Q_2^\eta(0) + A_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1^\eta(s)\wedge \eta n_s\right)ds\right) - A_{21}\left(\int_0^t \delta_s Q_2^\eta(s)ds\right)\\
&= Q_2^\eta(0) + A_{12}\left(\int_0^t \eta(1-p)\mu_s\left(\frac{1}{\eta}Q_1^\eta(s)\wedge n_s\right)ds\right) - A_{21}\left(\int_0^t \eta\delta_s\left(\frac{1}{\eta}Q_2^\eta(s)\right)ds\right).
\end{aligned}
$$

Fluid limits

By Theorem 2.2 (FSLLN) in [37]

$$\lim_{\eta \to \infty} \frac{Q^{\eta}(t)}{\eta} = Q^{(0)}(t) \qquad a.s.$$

where $Q^{(0)}(t)$ is called the *fluid approximation* and is the solution of the following ODE:

$$Q_1^{(0)}(t) = Q_1^{(0)}(0) + \int_0^t \left(\lambda_s - \mu_s \left(Q_1^{(0)}(s) \wedge n_s \right) + \delta_s Q_2^{(0)}(s) \right) ds$$
$$Q_2^{(0)}(t) = Q_2^{(0)}(0) + \int_0^t \left((1-p)\mu_s \left(Q_1^{(0)}(s) \wedge n_s \right) - \delta_s Q_2^{(0)}(s) \right) ds.$$

Special case – Fixed parameters

- The differential equations become:

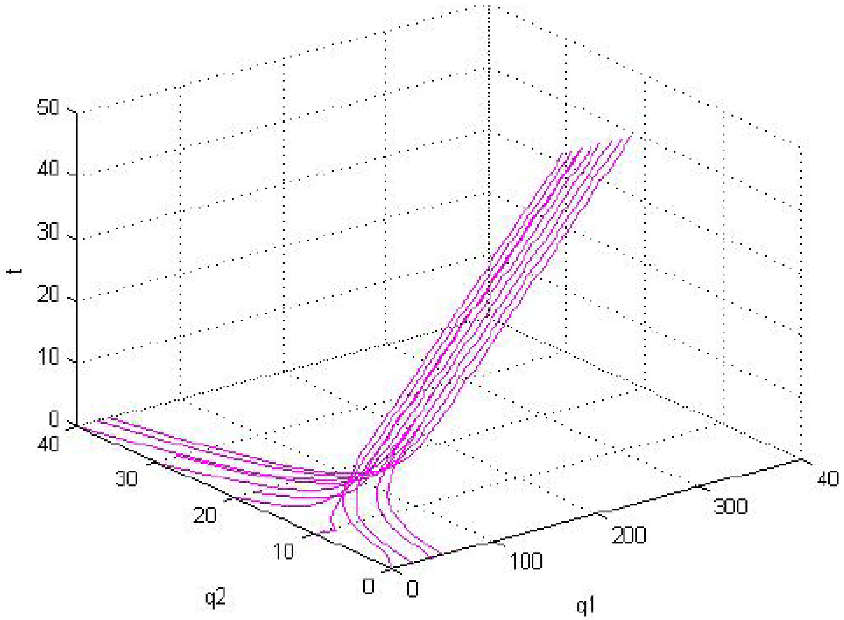
$$\frac{\partial q_1}{\partial t}(t) = \lambda + \delta q_2(t) - \mu(\bar{s} \wedge q_1(t))$$

$$\frac{\partial q_2}{\partial t}(t) = -\delta q_2(t) + p\mu(\bar{s} \wedge q_1(t))$$

- Steady-state analysis: What happens when t→∞?

Steady-state analysis

- Overloaded - $\bar{s} < \dfrac{\lambda}{(1-p)\mu}$ Underloaded - $\bar{s} > \dfrac{\lambda}{(1-p)\mu}$



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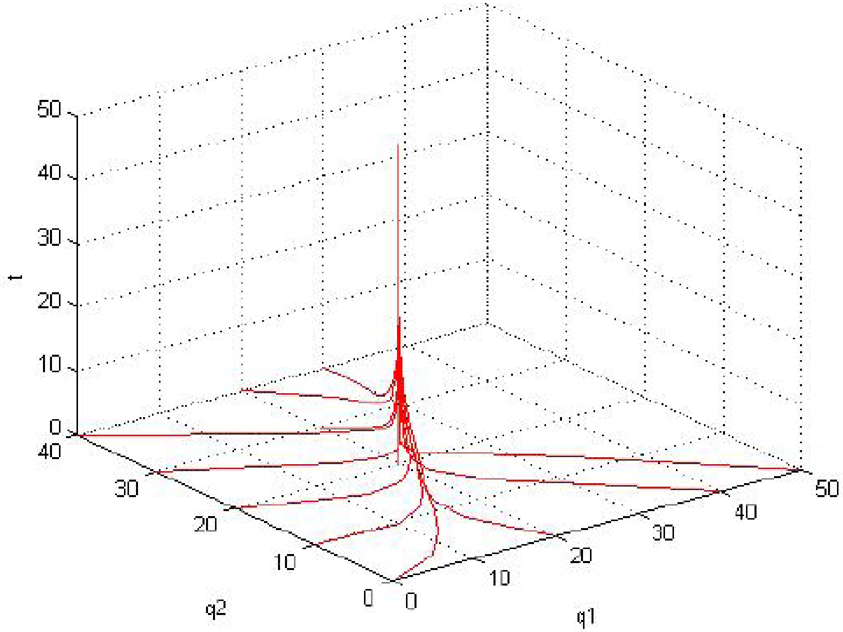
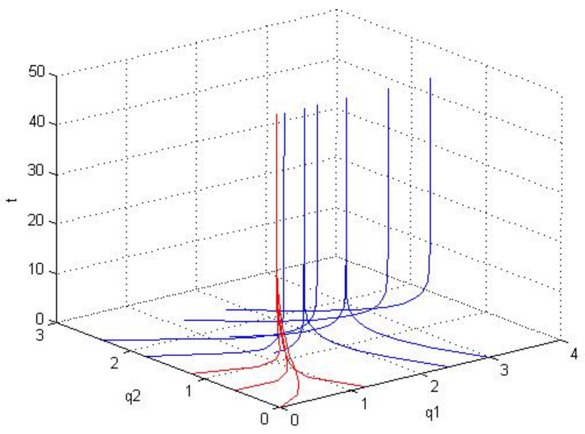
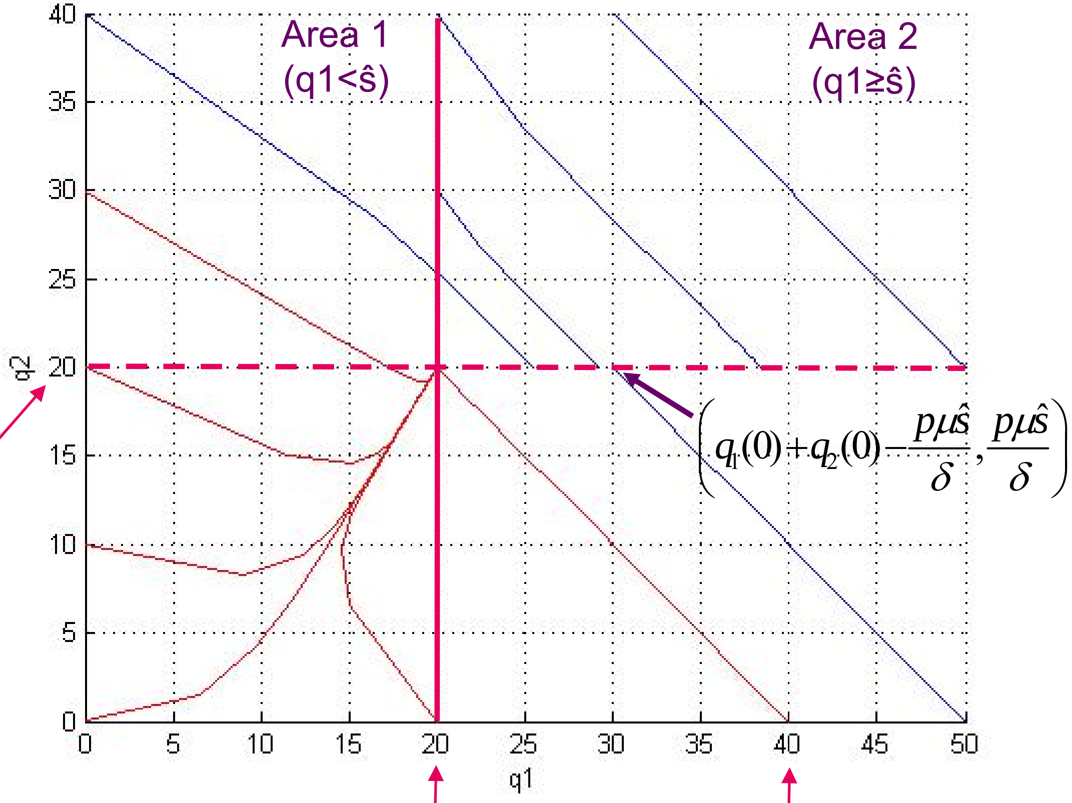


(underloaded)

Critically Loaded System: $\hat{s} = \dfrac{\lambda}{(1-p)\mu}$





$$\frac{p\mu\hat{s}}{\delta} = \frac{p\lambda}{(1-p)\delta}$$

$$\hat{s} = \frac{\lambda}{(1-p)\mu}$$

$$\hat{s} + \frac{p\mu\hat{s}}{\delta}$$

Steady-state analysis - Summery

$$\lim_{t\to\infty} q_1(t) = \begin{cases} q_2(0) - \frac{p\mu\bar{s}}{\delta} + q_1(0) & if \;\; \lambda - (1-p)\mu\bar{s} = 0, \\ \infty & if \;\; \lambda - (1-p)\mu\bar{s} > 0, \\ q_1(t_\infty) = \bar{s}, \; t_\infty < \infty & \text{otherwise.} \end{cases}$$

$$\lim_{t\to\infty} q_2(t) = \frac{p\mu\bar{s}}{\delta}.$$

Diffusion limits

Theorem 2.3 (FCLT) in [37]

$$\lim_{\eta\to\infty} \sqrt{\eta}\left[\frac{Q^{\eta}(t)}{\eta} - Q^{(0)}(t)\right] \stackrel{d}{=} Q^{(1)}(t) \tag{10.2}$$

where $Q^{(1)}(t)$ is called the *diffusion approximation* and is the solution of the following SDE (Stochastic Differential Equation):

$$\begin{aligned}
Q_1^{(1)}(t) &= Q_1^{(1)}(0) + \int_0^t \left(\mu_s 1_{\left\{Q_1^{(0)}(s)\le n_s\right\}} Q_1^{(1)}(s)^- - \mu_s 1_{\left\{Q_1^{(0)}(s)< n_s\right\}} Q_1^{(1)}(s)^+ + \delta_s Q_2^{(1)}(s)\right) ds \\
&\quad + B_1^a\left(\int_0^t \lambda_s ds\right) - B_2^d\left(\int_0^t p\mu_s\left(Q_1^{(0)}(s)\wedge n_s\right) ds\right) - B_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1^{(0)}(s)\wedge n_s\right) ds\right) \\
&\quad + B_{21}\left(\int_0^t \delta_s Q_2^{(0)}(s) ds\right), \\
Q_2^{(1)}(t) &= Q_2^{(1)}(0) + \int_0^t \left((1-p)\mu_s 1_{\left\{Q_1^{(0)}(s)< n_s\right\}} Q_1^{(1)}(s)^+ - (1-p)\mu_s 1_{\left\{Q_1^{(0)}(s)\le n_s\right\}} Q_1^{(1)}(s)^- - \delta_s Q_2^{(1)}(s)\right) ds \\
&\quad + B_{12}\left(\int_0^t (1-p)\mu_s\left(Q_1^{(0)}(s)\wedge n_s\right) ds\right) - B_{21}\left(\int_0^t \delta_s Q_2^{(0)}(s) ds\right),
\end{aligned} \tag{10.3}$$

where B_1^a, B_2^d, B_{12} and B_{21} are four mutually independent, standard (mean is 0 and the variance at time t is t) Brownian motions, $x^+ \equiv \max(x,0)$, and $x^- \equiv \max(-x,0) = -\min(x,0)$.

Diffusion limits

- Using the ODE one can find equations for:

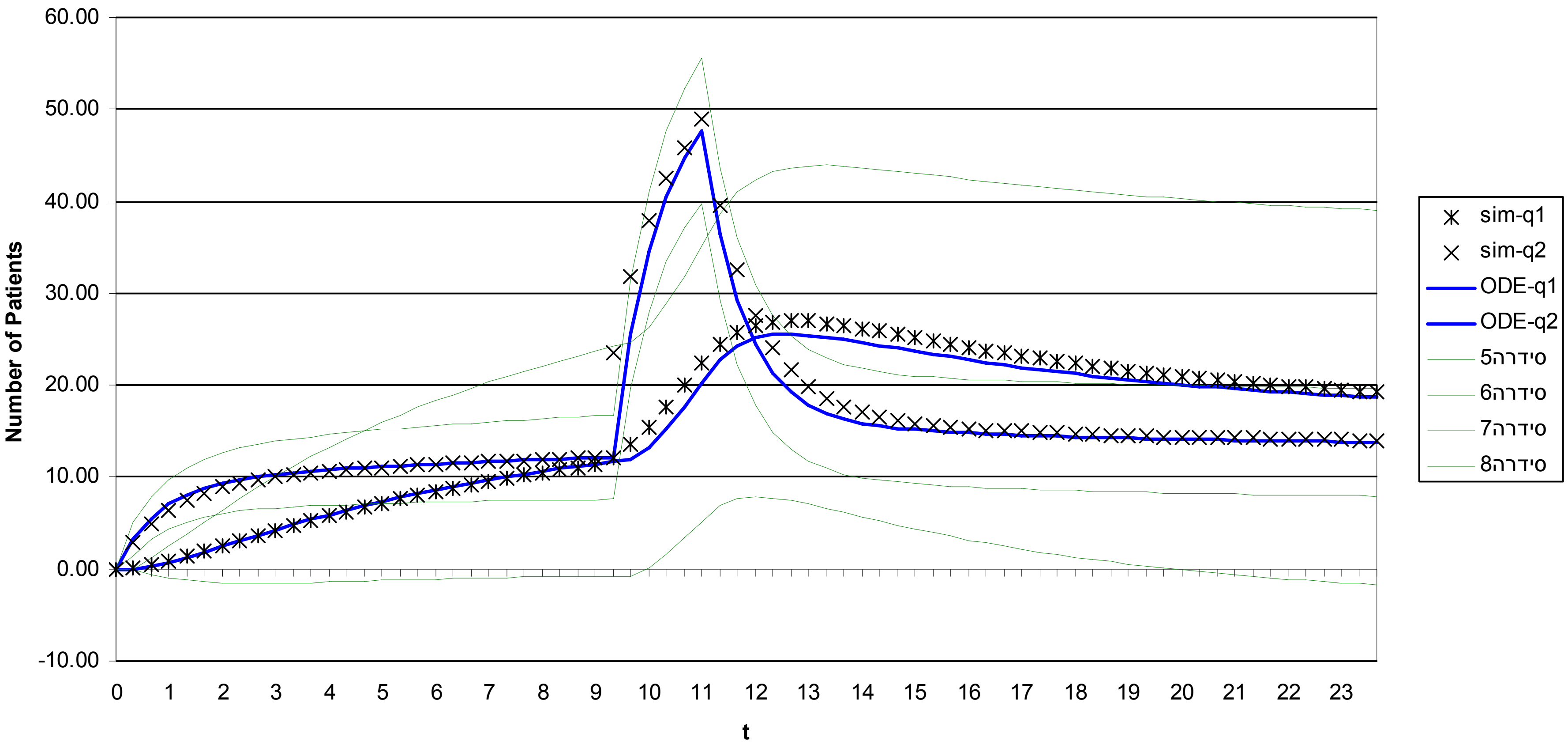
$$\frac{d}{dt}\mathrm{E}\left[Q_1^{(1)}(t)\right] \qquad \frac{d}{dt}\mathrm{Var}\left[Q_1^{(1)}(t)\right]$$

$$\frac{d}{dt}\mathrm{Cov}\left[Q_1^{(1)}(t), Q_2^{(1)}(t)\right]$$

$$\frac{d}{dt}\mathrm{E}\left[Q_2^{(1)}(t)\right] \qquad \frac{d}{dt}\mathrm{Var}\left[Q_2^{(1)}(t)\right]$$

- And to use them in analyzing time-variabil system. For example:

Delta = 0.2; Mu = 1; p = 0.25; s = 50; Lambda=10 (t<9 or t>11), Lambda=50 (9<t<11)



Future Research

- Investigating approximation of closed system
  - From which *n* are the approximations accurate? (simulation vs. rates of convergence)
- Optimization
  - Solving the bed-nurse optimization problem
  - Difference between hierarchical and simultaneous planning methods
- Validation of model using RFID data
- Expanding the model (Heterogeneous patients; adding doctors)


Thank you