STAT 991. Service Engineering. January 2005
The Wharton School. University of Pennsylvania.

PASTA = Poisson Arrivals See Time Averages

A rigorous proof in the following general setting was first presented by R.Wolff.

Arrivals (Observations):
$A = \{A(t),\ t \geq 0\}$,
$A(t)$ = number of arrivals in $[0, t]$.

System:
$X = \{X(t),\ t \geq 0\}$,
$X(t)$ = state at time t.

$$\textit{Time}\text{ average} \qquad \bar{\tau} \triangleq \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X(t)dt$$

$$\textit{Customer}\text{ average} \qquad \bar{c} \triangleq \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} X(S_n-)$$

where S_n = n-th arrival time.

PASTA assumptions:

(i) A is **Poisson**, and

(ii) **Lack of Anticipation.** For every $t \geq 0$, $\{A(t+u) - A(t) : u \geq 0\}$ is independent of $\{X(s) : 0 \leq s \leq t\}$.

Then $\bar{\tau} = \bar{c}$, in the following precise sense:
If one limit exists, then the other exists as well, in which case they are equal.

Second version: Let B be an arbitrary subset of a process X. Define by

$$\bar{\tau}_B \triangleq \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T I\{X(t) \in B\}dt$$

the fraction of time that the system spends in B, and let

$$\bar{c}_B \triangleq \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} I\{X(S_n-) \in B\}$$

be the fraction of arrivals that find ("see") the system in B.
Then $\bar{\tau}_B = \bar{c}_B$ in the same sense as above.

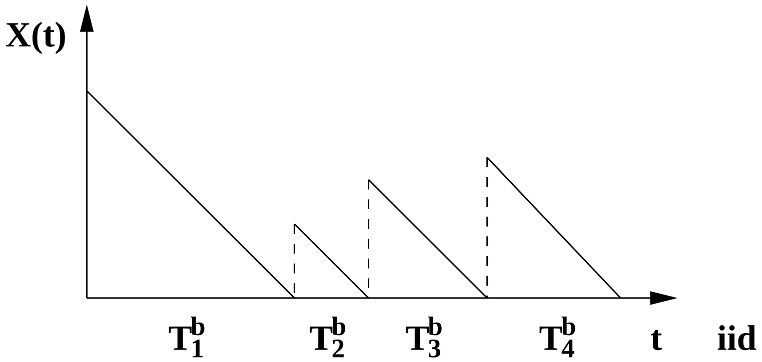
Application of **PASTA: Biased Sampling**

A *renewal process* is a counting process with iid interarrivals.

Descriptions:	$R = \{R(t),\ t \geq 0\}$ or	$\{T_1, T_2, \ldots\}$ iid, or	$\{S_1, S_2, \ldots\}$
Example:	Poisson	exponential	Erlang

Story: Buses arrive to a bus stop according to a renewal process $R_b = \{R_b(t), t \geq 0\}$.
T_i^b — times between arrivals of the buses.
Passengers arrive to the bus stop in a completely random fashion (Poisson).
S_i^p — arrival times of the passengers.

Question: How long, on average, do they wait? Plan service-level.



$A = \{A(t),\ t \geq 0\}$ = Poisson arrivals of passengers.

$X = \{X(t),\ t \geq 0\}$ = state = *Virtual waiting time.*

PASTA: $$\lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X(t)dt = \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} X(S_n^p -) = \bar{\tau}$$

$$\begin{aligned}
\Rightarrow \bar{\tau} &= \frac{1}{T} \cdot \ \text{(area under } X\text{, over } [0,T]) \\
&\approx \frac{1}{T} \cdot \left(\frac{1}{2}(T_1^b)^2 + \frac{1}{2}(T_2^b)^2 + \cdots + \frac{1}{2}(T_{R_b(T)}^b)^2 \right) \\
&= \frac{R_b(T)}{T} \cdot \frac{1}{2} \cdot \frac{T_1^2 + \cdots + T_{R_b(T)}^2}{R_b(T)} \xrightarrow[T\uparrow\infty]{} \frac{1}{E(T_1^b)} \cdot \frac{1}{2} \cdot E(T_1^b)^2 \text{, by SLLN} \\
&= \underbrace{\frac{1}{2} E(T_1^b)}_{\text{“Deterministic” answer}} \qquad \underbrace{[1 + c^2(T_1^b)]}_{\text{Bias, due to variability}}, \qquad c = \frac{\sigma}{E} \text{ coefficient of variation.}
\end{aligned}$$

Check Poisson bus arrivals to derive Paradox:
1(“stochastic” answer) $= \frac{1}{2}$ (“deterministic” answer).