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Call Centers with Impatient Customers: Exact Analysis and Many-Server Asymptotics of the M/M/n+G Queue

Ph.D. seminar

Supervisor: Professor Avishai Mandelbaum

Based on:

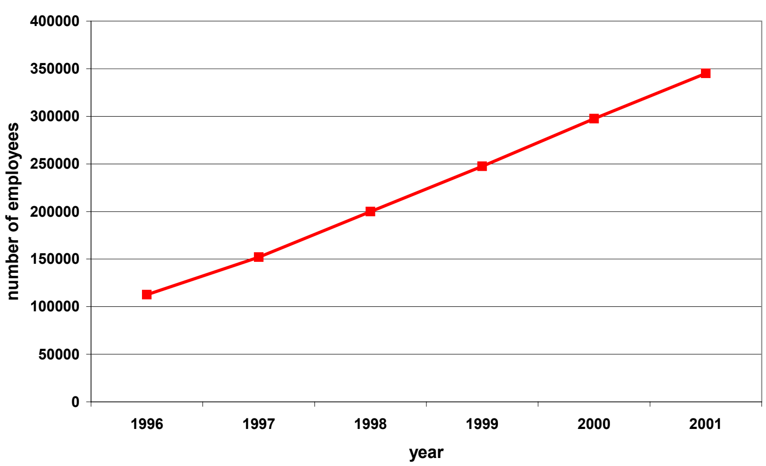
- The Impact of Customers' Patience on Delay and Abandonment: Some Empirically-Driven Experiments with the M/M/n+G Queue, *OR Spectrum* (2004) 26:377-411.
- Call Centers with Impatient Customers: Many-Server Asymptotics of the M/M/n+G Queue. To be submitted to *Management Science*.
- Lecture notes for the course *Service Engineering*.

The World of Call Centers

Israel: 500 call centers, not including tourism, medical care, emergency.
11,000 agents: 50% – service, 25% – information,
25% – tele-marketing. (Rafaeli, 2004)

U.S. 3% workforce (several millions);
1000's agents in a "single" call center.
Growing extensively:

Germany: number of call center employees



Quality of Service

- **Accessibility** of agents;
- Effectiveness of service;
- Customer-agent interactions.

Efficiency of Service

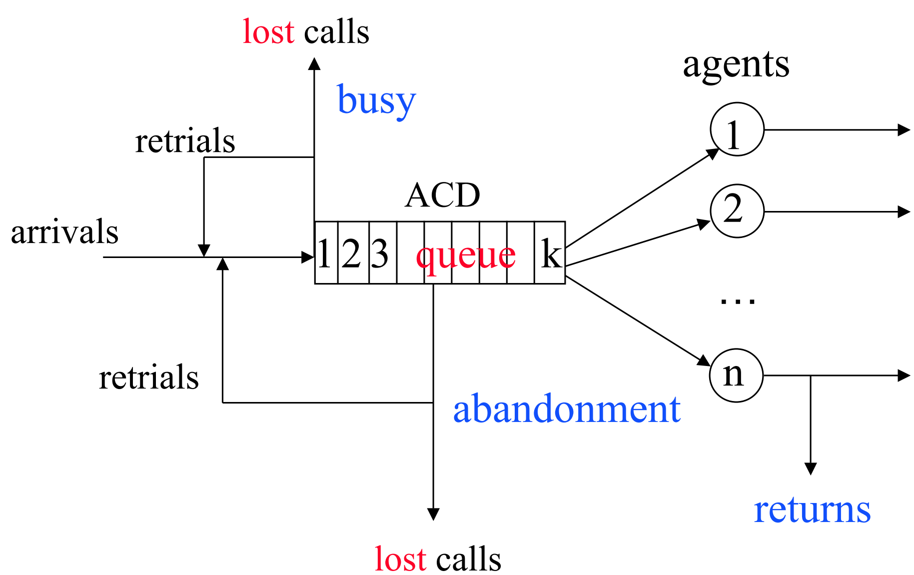
- Yearly & monthly level: hiring and training;
- Weekly & daily level: **queueing** and scheduling.

Quality/Efficiency Tradeoff:

having the right number of agents in place at the right times.

Modelling a Call Center.

Schematic representation of a basic telephone call center



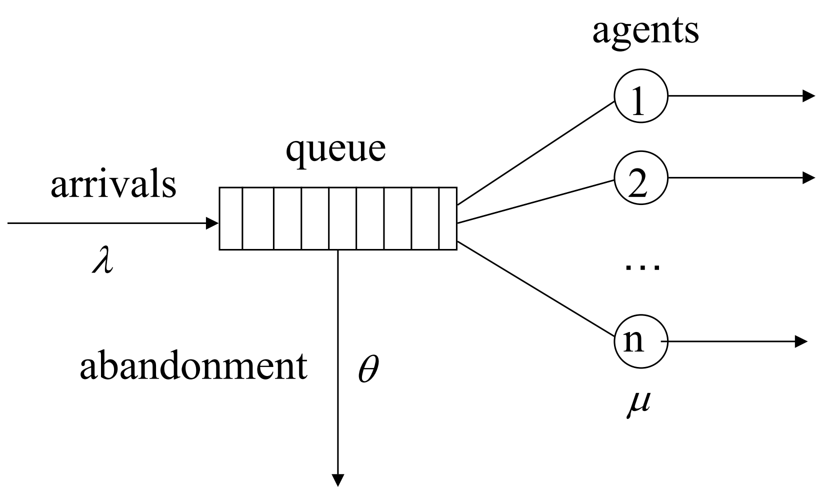
M/M/n (Erlang-C) – prevalent model (still):

- Homogeneous Poisson arrivals, rate λ;
- Exponential service, rate μ, mean $\mathrm{E}[S]$;
- n service agents.

Patience in Invisible Queues

Why ignoring abandonment is bad?

- One of a few customer-subjective performance measures;
- Distorted Service Level definitions – $\mathrm{P}\{W < T|\text{Served}\}$, ASA;
- Wrong staffing calculations.



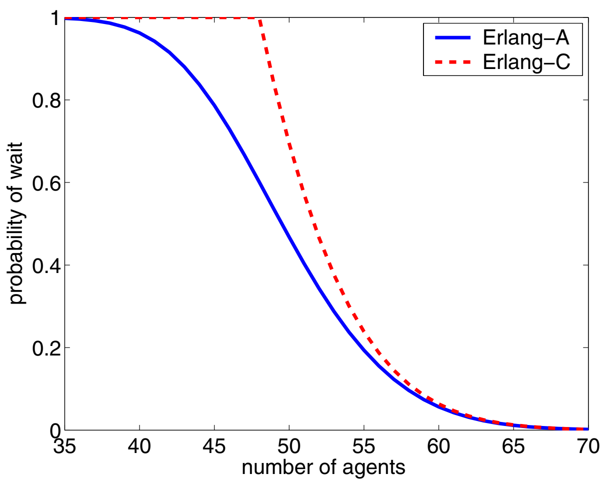
M/M/n+M (Erlang-A, Palm) – simplest model with abandonment, used by well-run call centers.

- **Patience time** $\tau \sim \exp(\theta)$:
  time a customer is willing to wait for service;
- **Offered wait** V:
  waiting time of a customer with infinite patience;
- If $\tau \leq V$, customer abandons; otherwise, gets service;
- **Actual wait** $W = \min(\tau, V)$.

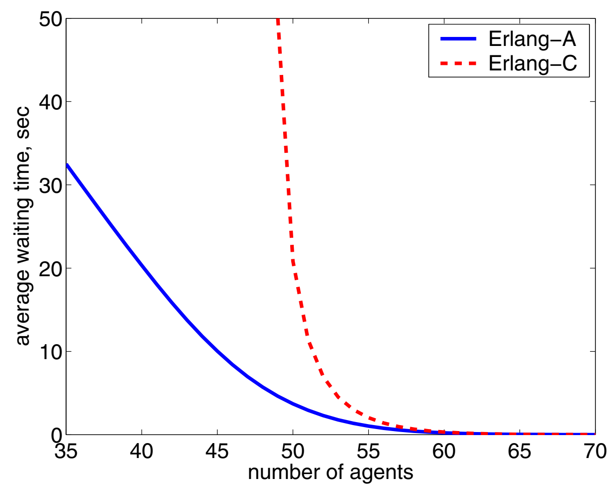
Erlang-A vs. Erlang-C

48 calls per min, 1 min average service time,
2 min average patience

probability of wait
vs. number of agents



average wait
vs. number of agents



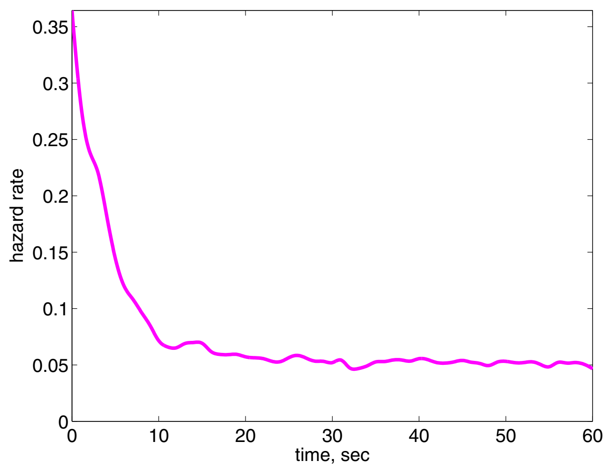
If 50 agents:

	M/M/n	M/M/n+M	M/M/n, $\lambda \downarrow 3.1\%$
Fraction abandoning	–	3.1%	-
Average waiting time	20.8 sec	3.7 sec	8.8 sec
Waiting time's 90-th percentile	58.1 sec	12.5 sec	28.2 sec
Average queue length	17	3	7
Agents' utilization	96%	93%	93%

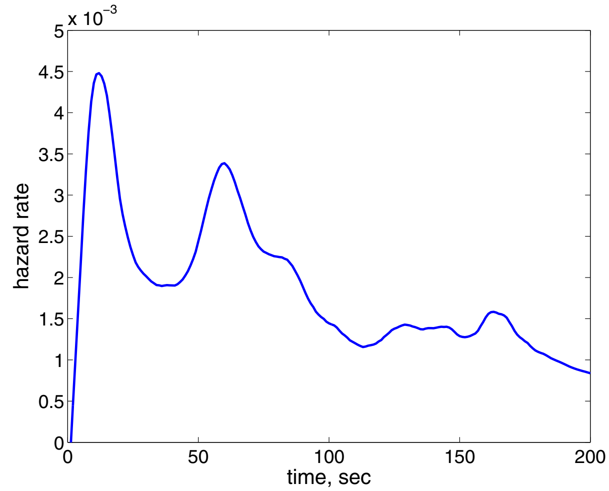
Effect of Patience Distribution

Are patience times really exponential? **Negative** examples:

US bank



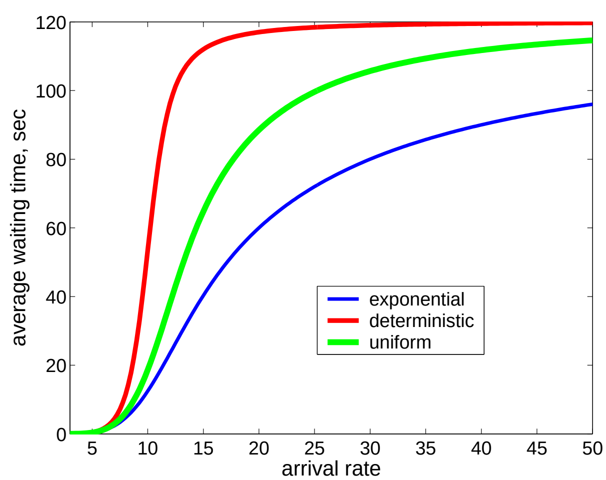
Israeli bank



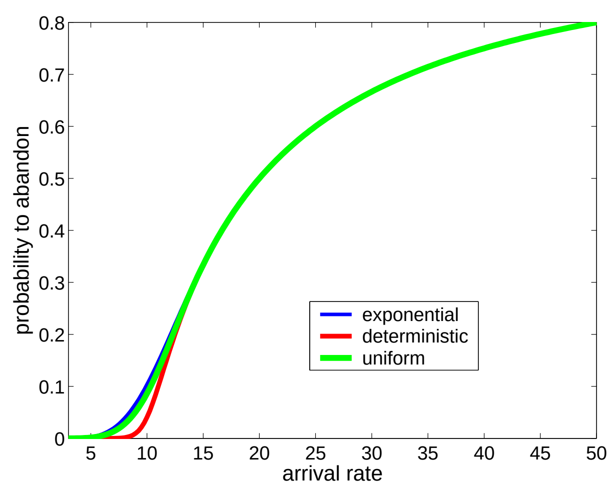
Does distribution of patience times affect system performance?

1 min average service time, 2 min average patience, 10 agents, arrival rate varies from 3 to 50 per minute

average wait



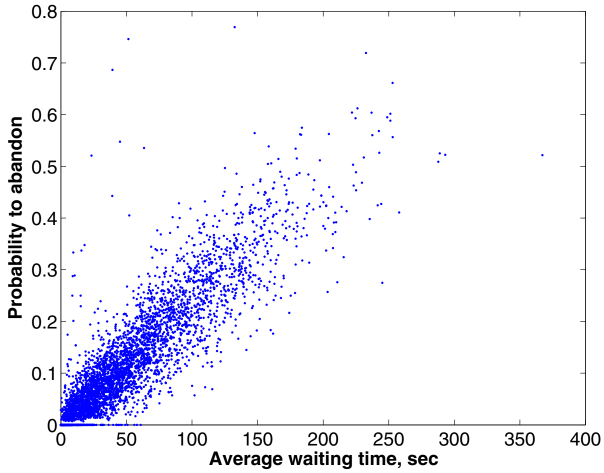
probability to abandon

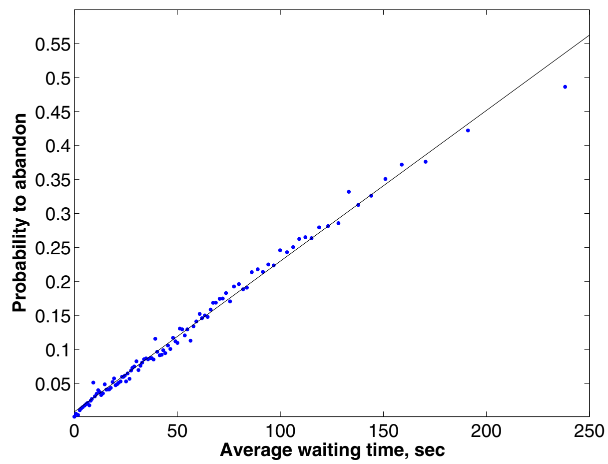


Conclusion: study models with general patience.

On the Relation between P{Ab} and E[*W*]

Yearly Call Center data: linear pattern

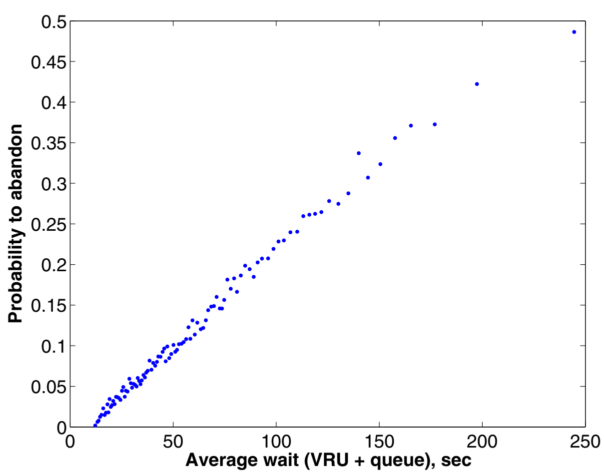




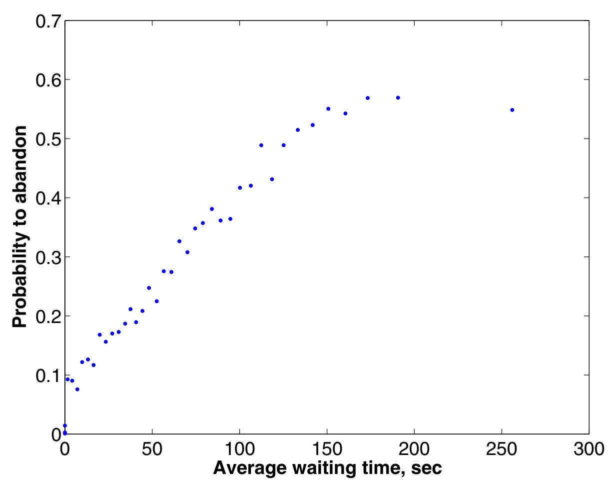
The graphs are based on 4158 hour intervals.

Linear patterns with non-zero intercepts

VRU-time included in wait



New customers



If Patience is $\exp(\theta)$, then

$$\mathrm{P}\{\mathrm{Ab}\} \;=\; \theta \cdot \mathrm{E}[W]\,.$$

(Proof: based on Little's Law + conservation $\lambda \cdot \mathrm{P}\{\mathrm{Ab}\} = \theta \cdot \mathrm{E}[Q]$.)

Operational Performance Measures

The most popular performance measure is $P\{W \le T; \mathrm{Sr}\}$ or even $P\{W \le T \,|\, \mathrm{Sr}\}$.

We recommend either:

- $P\{W \le T; \mathrm{Sr}\}$ - fraction of well-served;
- $P\{\mathrm{Ab}\}$ - fraction of poorly-served.

or four-dimensional refinement:

- $P\{W \le T; \mathrm{Sr}\}$ - fraction of well-served;
- $P\{W > T; \mathrm{Sr}\}$ - fraction of served, with a potential for improvement (say, a higher priority on next visit);
- $P\{W > \epsilon; \mathrm{Ab}\}$ - fraction of poorly-served;
- $P\{W \le \epsilon; \mathrm{Ab}\}$ - fraction of those whose service-level is undetermined.

M/M/n+G Queue

- λ – Poisson arrival rate.
- μ – Exponential service rate.
- n service agents.
- G – Patience distribution.

Exact results:

- Baccelli and Hebuterne (1981) – probability to abandon, distribution of offered wait:
- Brandt and Brandt (1999, 2002) – number-in-system and waiting time distributions.
- Mandelbaum, Zeltyn (2004) – extensive list of performance measures.

Research goals:

M/M/n+G research:

- Quality/efficiency tradeoff;
- Asymptotic analysis of moderate-to-large call centers;
- Impact of patience distribution on P{Ab}/E[W] relation and performance measures.

M/M/n+G Queue: Calculation of Performance Measures

Building blocks:

$$H(x) \triangleq \int_0^x \bar{G}(u)du\,,$$

where $\bar{G}(\cdot)$ is survival function of patience time.

$$\begin{aligned}
J &\triangleq \int_0^\infty \exp\{\lambda H(x) - n\mu x\}\, dx\,,\\
J_1 &\triangleq \int_0^\infty x \cdot \exp\{\lambda H(x) - n\mu x\}\, dx\,,\\
J_H &\triangleq \int_0^\infty H(x) \cdot \exp\{\lambda H(x) - n\mu x\}\, dx\,,\\
J(t) &\triangleq \int_t^\infty \exp\{\lambda H(x) - n\mu x\}\, dx\,.\\
J_1(t) &\triangleq \int_t^\infty x \cdot \exp\{\lambda H(x) - n\mu x\}\, dx\,,\\
J_H(t) &\triangleq \int_t^\infty H(x) \cdot \exp\{\lambda H(x) - n\mu x\}\, dx\,.
\end{aligned}$$

Finally,

$$\mathcal{E} \triangleq \frac{\sum_{j=0}^{n-1} \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j}{\frac{1}{(n-1)!}\left(\frac{\lambda}{\mu}\right)^{n-1}}\,.$$

Performance measures calculated via building blocks:

P{Ab} – probability to abandon, P{Sr} – probability to be served,
W – waiting time, V – offered wait,
Q – queue length.

$$\begin{aligned}
\mathrm{P}\{V>0\} &= \frac{\lambda J}{\mathcal{E}+\lambda J}\,, \\
\mathrm{P}\{W>0\} &= \frac{\lambda J}{\mathcal{E}+\lambda J}\cdot \bar{G}(0)\,, \\
\mathrm{P}\{\mathrm{Ab}\} &= \frac{1+(\lambda-n\mu)J}{\mathcal{E}+\lambda J}\,, \\
\mathrm{P}\{\mathrm{Sr}\} &= \frac{\mathcal{E}+n\mu J-1}{\mathcal{E}+\lambda J}\,, \\
\mathrm{E}[V] &= \frac{\lambda J_1}{\mathcal{E}+\lambda J}\,, \\
\mathrm{E}[W] &= \frac{\lambda J_H}{\mathcal{E}+\lambda J}\,, \\
\mathrm{E}[Q] &= \frac{\lambda^2 J_H}{\mathcal{E}+\lambda J}\,, \\
\mathrm{E}[W \mid \mathrm{Ab}] &= \frac{J+\lambda J_H-n\mu J_1}{(\lambda-n\mu)J+1}\,, \\
\mathrm{E}[W \mid \mathrm{Sr}] &= \frac{n\mu J_1-J}{\mathcal{E}+n\mu J-1}\,, \\
\mathrm{P}\{W>t\} &= \frac{\lambda\bar{G}(t)J(t)}{\mathcal{E}+\lambda J}\,, \\
\mathrm{E}[W \mid W>t] &= \frac{J_H(t)-(H(t)-t\bar{G}(t))\cdot J(t)}{\bar{G}(t)J(t)}\,, \\
\mathrm{P}\{\mathrm{Ab} \mid W>t\} &= \frac{\lambda-n\mu-G(t)}{\lambda\bar{G}(t)}+\frac{\exp\{\lambda H(t)-n\mu t\}}{\lambda\bar{G}(t)J(t)}\,.
\end{aligned}$$

Asymptotic Operational Regimes

Example of Half-Hour ACD Report

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
Total	20,577	19,860	3.5%	30	307	95.1%	
8:00	332	308	7.2%	27	302	87.1%	59.3
8:30	653	615	5.8%	58	293	96.1%	104.1
9:00	866	796	8.1%	63	308	97.1%	140.4
9:30	1,152	1,138	1.2%	28	303	90.8%	211.1
10:00	1,330	1,286	3.3%	22	307	98.4%	223.1
10:30	1,364	1,338	1.9%	33	296	99.0%	222.5
11:00	1,380	1,280	7.2%	34	306	98.2%	222.0
11:30	1,272	1,247	2.0%	44	298	94.6%	218.0
12:00	1,179	1,177	0.2%	1	306	91.6%	218.3
12:30	1,174	1,160	1.2%	10	302	95.5%	203.8
13:00	1,018	999	1.9%	9	314	95.4%	182.9
13:30	**1,061**	**961**	**9.4%**	**67**	**306**	**100.0%**	**163.4**
14:00	1,173	1,082	7.8%	78	313	99.5%	188.9
14:30	**1,212**	**1,179**	**2.7%**	**23**	**304**	**96.6%**	**206.1**
15:00	1,137	1,122	1.3%	15	320	96.9%	205.8
15:30	1,169	1,137	2.7%	17	311	97.1%	202.2
16:00	1,107	1,059	4.3%	46	315	99.2%	187.1
16:30	914	892	2.4%	22	307	95.2%	160.0
17:00	**615**	**615**	**0.0%**	**2**	**328**	**83.0%**	**135.0**
17:30	420	420	0.0%	0	328	73.8%	103.5
18:00	49	49	0.0%	14	180	84.2%	5.8

Asymptotic Operational Regimes

Efficiency-Driven (ED) regime

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
13:30	**1,061**	**961**	**9.4%**	**67**	**306**	**100.0%**	**163.4**

- 100% occupancy;
- high P{Ab};
- considerable ASA;
- $\mathbf{P\{W > 0\} \approx 1}$.

Offered load

$$R_{ED} \triangleq \frac{\lambda}{\mu} = 1061 : \frac{1800}{306} = 180.37\,.$$

Definition:

$$n = R_{ED} \cdot (1 - \gamma) \qquad \gamma > 0.$$

In our case, *service grade*

$$\gamma = 1 - \frac{n}{R_{ED}} = 1 - \frac{163.4}{180.37} = 0.094 \approx \mathrm{P}\{\mathrm{Ab}\}\,.$$

- This case is similar to traditional queues in heavy traffic;
- See recent papers of Whitt (2004).

Quality-Driven (QD) regime

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
17:00	**615**	**615**	**0.0%**	**2**	**328**	**83.0%**	**135.0**

- Occupancy far below 100%;
- negligible P{Ab};
- very small ASA;
- $\mathbf{P\{W > 0\} \approx 0}$.

Offered load

$$R_{QD} \;=\; \frac{\lambda}{\mu} \;=\; 615 : \frac{1800}{328} \;=\; 112.07\,.$$

Definition:

$$n \;=\; R_{QD} \cdot (1+\gamma) \qquad \gamma > 0.$$

Service grade

$$\gamma \;=\; \frac{n}{R_{QD}} - 1 \;=\; \frac{135}{112.07} - 1 \;=\; 0.205\,.$$

Quality and Efficiency-Driven (QED) regime

Time	Calls	Answered	Abandoned%	ASA	AHT	Occ%	# of agents
14:30	**1,212**	**1,179**	**2.7%**	**23**	**304**	**96.6%**	**206.1**

- High occupancy, but not 100%;
- small P{Ab} and ASA;
- $\mathbf{P\{W > 0\} \approx \boldsymbol{\alpha}}$, $\quad 0 < \alpha < 1$.

$$R_{QED} \;=\; \frac{\lambda}{\mu} \;=\; 1212 : \frac{1800}{304} \;=\; 204.69\,.$$

Definition:

$$n \;=\; R_{QED} + \beta\sqrt{R_{QED}}\,, \qquad -\infty < \beta < \infty\,.$$

Service grade

$$\beta \;=\; \frac{n - R_{QED}}{\sqrt{R_{QED}}} = \frac{206.1 - 204.69}{\sqrt{204.69}} \;=\; 0.10\,.$$

Square-Rule Safety Staffing: Described by Erlang in 1924! Formal analysis:

- Erlang-C: Halfin & Whitt (1981), $\beta > 0$;
- Erlang-B (M/M/n/n): Jagerman (1974);
- Erlang-A: Garnett, Mandelbaum, Reiman (2002);
- M/M/n+G: Present thesis.

M/M/n+G: QED Operational Regime.

Main case: positive density of patience at the origin.

Density of patience time: $g = \{g(x), x \geq 0\}$, where $g(0) \stackrel{\Delta}{=} g_0 > 0$.

Fix service rate μ.

Let arrival rate $\lambda \to \infty$ and

$$n \;=\; \frac{\lambda}{\mu} + \beta\sqrt{\frac{\lambda}{\mu}} + o(\sqrt{\lambda}), \qquad -\infty < \beta < \infty.$$

Building blocks:

$$J \;=\; \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{\mu g_0}} \cdot \frac{1}{h(\hat{\beta})} + o\left(\frac{1}{\sqrt{n}}\right) \,,$$

$$\mathcal{E} \;=\; \frac{\sqrt{n}}{h(-\beta)} + o(\sqrt{n}) \,,$$

$$J_1 \;=\; \frac{1}{n\mu g_0}\left[1 - \frac{\hat{\beta}}{h(\hat{\beta})}\right] + o\left(\frac{1}{n}\right) \,,$$

where

$$\hat{\beta} \;\stackrel{\Delta}{=}\; \beta\sqrt{\frac{\mu}{g_0}} \,,$$

$h(\cdot)$ – hazard rate of standard normal distribution.

Proofs: Combine M/M/n+G formulae above and the Laplace method for asymptotic calculation of integrals.

Main case: performance measures

- Probability of wait converges to constant:

$$\mathrm{P}\{\mathrm{W} > 0\} \;\sim\; \left[1 + \sqrt{\frac{g_0}{\mu}} \cdot \frac{h(\hat{\beta})}{h(-\beta)}\right]^{-1} .$$

- Probability to abandon decreases at rate $\frac{1}{\sqrt{n}}$:

$$\mathrm{P}\{\mathrm{Ab}|W > 0\} \;=\; \frac{1}{\sqrt{n}} \cdot \sqrt{\frac{g_0}{\mu}} \cdot \left[h(\hat{\beta}) - \hat{\beta}\right] + o\left(\frac{1}{\sqrt{n}}\right) .$$

- Average wait decreases at rate $\frac{1}{\sqrt{n}}$:

$$\mathrm{E}[W|W > 0] \;=\; \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{g_0\mu}} \cdot \left[h(\hat{\beta}) - \hat{\beta}\right] + o\left(\frac{1}{\sqrt{n}}\right) .$$

- Ratio between $\mathrm{P}\{\mathrm{Ab}\}$ and $\mathrm{E}[W]$ converges to patience density at the origin:

$$\boxed{\frac{\mathrm{P}\{\mathrm{Ab}\}}{\mathrm{E}[W]} \;\sim\; g_0}$$

- Asymptotic distribution of wait:

$$\mathrm{P}\left\{\frac{W}{\mathrm{E}[S]} > \frac{t}{\sqrt{n}} \,\middle|\, W > 0\right\} \;\sim\; \frac{\bar{\Phi}\left(\hat{\beta} + \sqrt{\frac{g_0}{\mu}} \cdot t\right)}{\bar{\Phi}(\hat{\beta})}, \qquad t \geq 0 .$$

- Probability to abandon given delay in queue

$$\mathrm{P}\left\{\mathrm{Ab} \,\middle|\, \frac{W}{\mathrm{E}[S]} > \frac{t}{\sqrt{n}}\right\} \;=\; \frac{1}{\sqrt{n}} \cdot \sqrt{\frac{g_0}{\mu}} \cdot \left[h\left(\hat{\beta} + t\sqrt{\frac{g_0}{\mu}}\right) - \hat{\beta}\right] + o\left(\frac{1}{\sqrt{n}}\right) .$$

QED Operational Regime: Discussion

Points of view.

- **Customers:** $\mathrm{P}\{W > 0\} \approx \alpha, \quad \mathrm{P}\{\mathrm{Ab}\} \approx \dfrac{\gamma}{\sqrt{n}}$;
- **Agents:** Offered load per Server $= \dfrac{R}{n} \approx 1 - \dfrac{\beta}{\sqrt{n}}$;
- **Managers:** $n \approx R + \beta\sqrt{R}$.

$\boldsymbol{\beta = 0}$: right answer for wrong reasons.

(Common in stochastic-ignorant operations.)

If $\beta = 0$, QED staffing level:

$$n = \frac{\lambda}{\mu} \; = \; R\,.$$

Equivalent to deterministic rule: assign number of agents equal to *offered load*.

Erlang-C: queue "explodes".

M/M/n+G: assume $\mu = \theta$. Then $\mathrm{P}\{W = 0\} \approx 50\%$.

If $n = 100$, $\mathrm{P}\{\mathrm{Ab}\} \approx 4\%$, and $\mathrm{E}[W] \approx 0.04 \cdot \mathrm{E}[S]$.

Overall, good service level.

QED Operational Regime: Special Cases

According to patience distribution.

- **Patience density vanishing near the origin.**
  (k-1) derivatives at the origin are zero, the k-th derivative is positive.
  Examples: Erlang, Phase-type.
  - If $\beta > 0$, wait similar to Erlang-C. P$\{$Ab$\}$ decreases at $n^{-(k+1)/2}$ rate.
  - If $\beta < 0$, almost all customers delayed, E$[W] \to 0$ slowly. P$\{$Ab$\} \approx -\beta/\sqrt{n}$.
  - If $\beta = 0$, intermediate behavior.
- **Delayed distribution of patience.**
  Customers do not abandon till $c > 0$.
  Examples: Delayed exponential, deterministic.
  Similar to the previous case. For $\beta < 0$, wait converges to c.
- **Balking.**
  Customer, not served immediately, balks with probability P$\{$Blk$\}$.
  Example. M/M/n/n (Erlang-B).
  - P$\{W > 0\}$ decreases at rate $1/\sqrt{n}$;
  - P$\{$Ab$|V > 0\} \approx$ P$\{$Blk$\}$;
  - P$\{$Ab$\} \approx h(-\beta)/\sqrt{n}$, asymptotic loss probability for Erlang-B.
- **Scaled balking.**
  Customer, not served immediately, balks with probability $p_b/\sqrt{n}$.
  Results are similar to the main case.

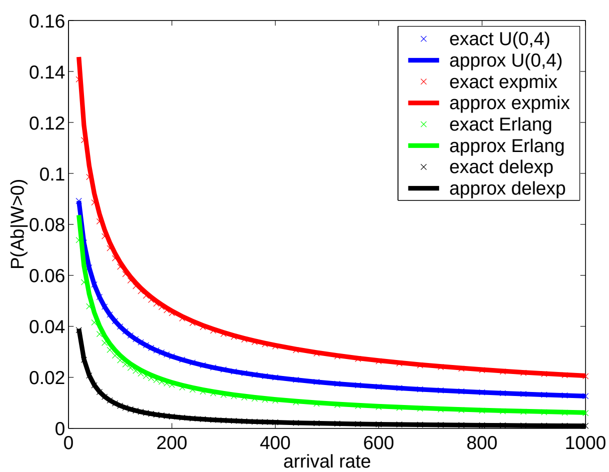
QED Regime: Numerical Experiments–1

Patience distributions:

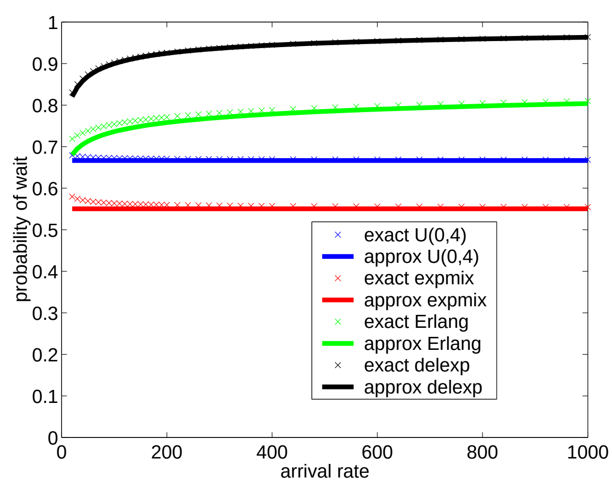
- *Uniform* on [0,4], $g_0 = 0.25$;
- *Hyperexponential*, 50-50% mixture of exp(mean=1) and exp(mean=1/3), $g_0 = 2/3$;
- *Erlang*, two exp(mean=1) phases, $g_0 = 0$;
- *Delayed exponential*, 1 + exp(mean=1), $g_0 = 0$.

Service grade $\beta = 0$.

Probability to abandon given delay vs. arrival rate



Probability of wait vs. arrival rate

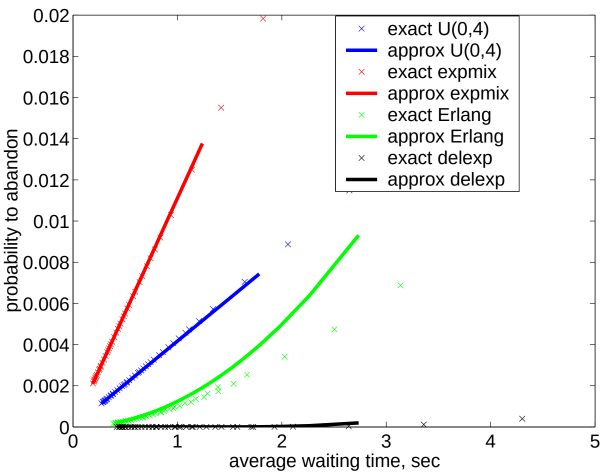


P{Ab} convergence rates: $1/\sqrt{n}$, $1/\sqrt{n}$, $n^{-2/3}$, exp, respectively.

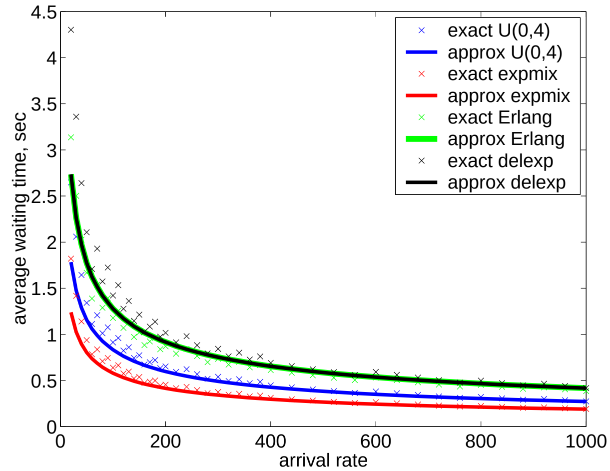
QED Regime: Numerical Experiments–2

Service grade $\beta = 1$.

Probability to abandon vs. average waiting time



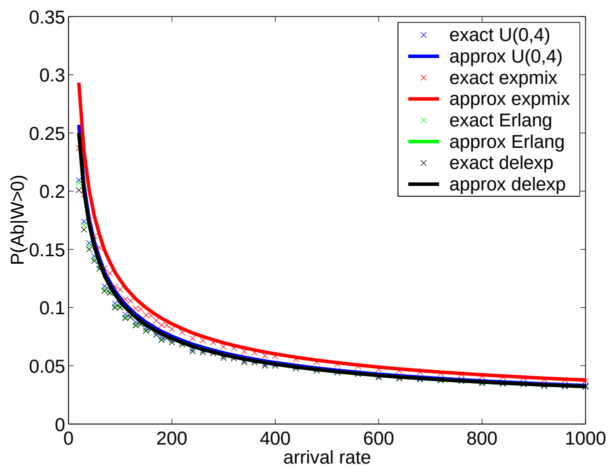
Average waiting time vs. arrival rate



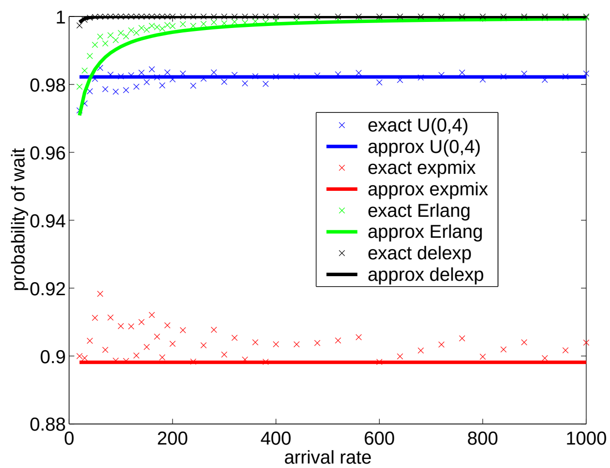
Note linear patterns in the first plot.

Service grade $\beta = -1$.

Probability to abandon given delay vs. arrival rate



Probability of wait vs. arrival rate



Convergence to $-\beta/\sqrt{n}$ for probability to abandon.

M/M/n+G: QD Operational Regime.

Density of patience time at the origin $g_0 > 0$.
Staffing level

$$n = \frac{\lambda}{\mu} \cdot (1+\gamma) + o(\sqrt{\lambda})\,, \qquad \gamma > 0\,.$$

Performance measures

- $\mathrm{P}\,\{W > 0\}$ decreases exponentially on n.
- Probability to abandon of delayed customers:

$$\mathrm{P}\{\mathrm{Ab}|W > 0\} \;=\; \frac{1}{n} \cdot \frac{1+\gamma}{\gamma} \cdot \frac{g_0}{\mu} + o\left(\frac{1}{n}\right)\,.$$

- Average wait of delayed customers:

$$\mathrm{E}[W \mid W > 0] \;=\; \frac{1}{n} \cdot \frac{1+\gamma}{\gamma} \cdot \frac{1}{\mu} + o\left(\frac{1}{n}\right)\,.$$

- Linear relation between $\mathrm{P}\{\mathrm{Ab}\}$ and $\mathrm{E}[W]$.

$$\boxed{\frac{\mathrm{P}\{\mathrm{Ab}\}}{\mathrm{E}[W]} \;\sim\; g_0}$$

- Asymptotic distribution of wait:

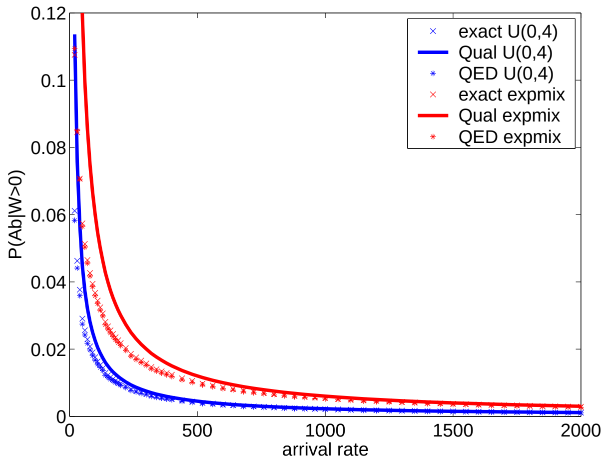
$$\mathrm{P}\left\{ \frac{W}{\mathrm{E}(S)} > \frac{t}{n} \;\middle|\; W > 0 \right\} \;\sim\; e^{-(1-\rho)t}\,, \qquad \rho = \frac{\lambda}{n\mu}\,.$$

QD Regime: Numerical Experiments

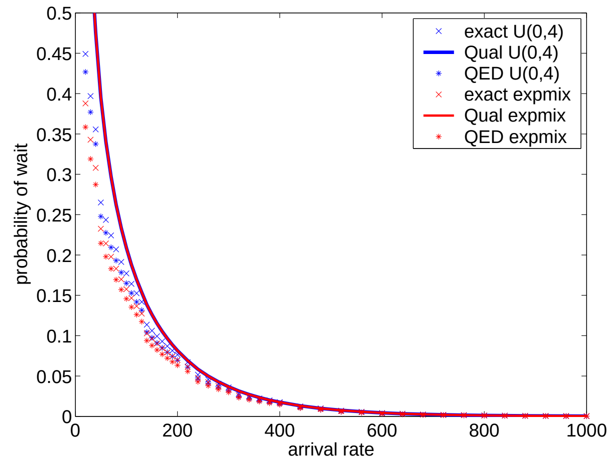
Patience distributions: Uniform, hyperexponential.

Service grade $\gamma = 1/9$, $\rho = 0.9$.

Probability to abandon given delay



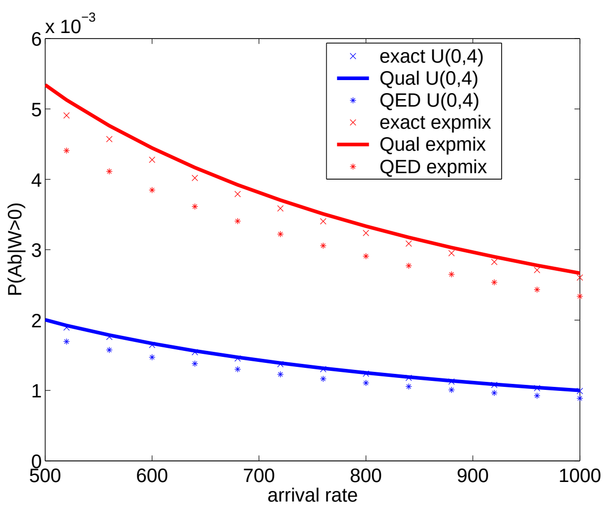
Probability of wait



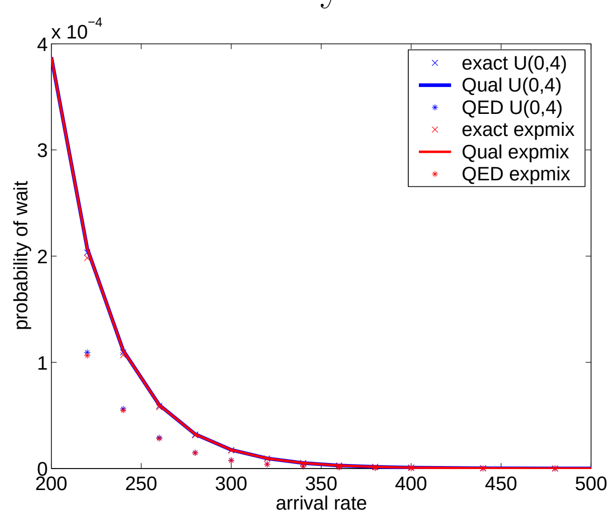
Overall, QED approximations are better than QD.

Service grade $\gamma = 0.25$, $\rho = 0.8$. Large arrival rate.

Probability to abandon given delay



Probability of wait



M/M/n+G: ED Operational Regime.

Assume $G(x) = \gamma$ has a unique solution x^* and $g(x^*) > 0$.

Staffing level

$$n = \frac{\lambda}{\mu} \cdot (1-\gamma) + o(\sqrt{\lambda}), \qquad \gamma > 0 .$$

Performance measures

- $\mathrm{P}\{W = 0\}$ decreases exponentially on n.
- Probability to abandon converges to:

$$\mathrm{P}\{\mathrm{Ab}\} \ \sim \ \gamma \ \approx \ 1 - \frac{1}{\rho} .$$

- Offered wait converges to x^*:

$$\mathrm{E}[V] \ \sim \ x^* , \qquad V \ \xrightarrow{p} \ x^* .$$

- Distribution G^* of $\min(x^*, \tau)$

$$G^*(x) \ = \ \begin{cases} G(x)/\gamma, & x \le x^* \\ 1, & x > x^* \end{cases}$$

  Asymptotic distribution of wait:

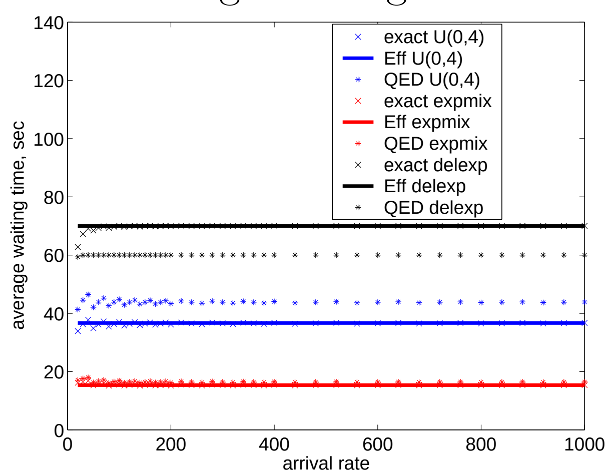
$$W \ \xrightarrow{w} \ G^* , \qquad \mathrm{E}[W] \ \to \ \mathrm{E}[\min(x^*, \tau)] .$$

ED Regime: Numerical Experiments

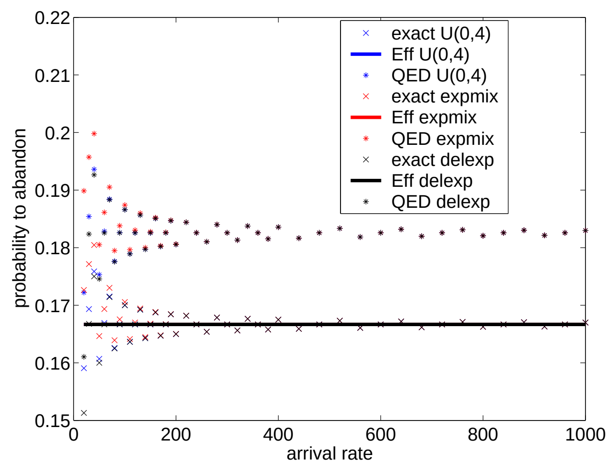
Patience distributions: Uniform, hyperexponential, delayed exponential.

Service grade $\gamma = 1/6$, $\rho = 1.2$.

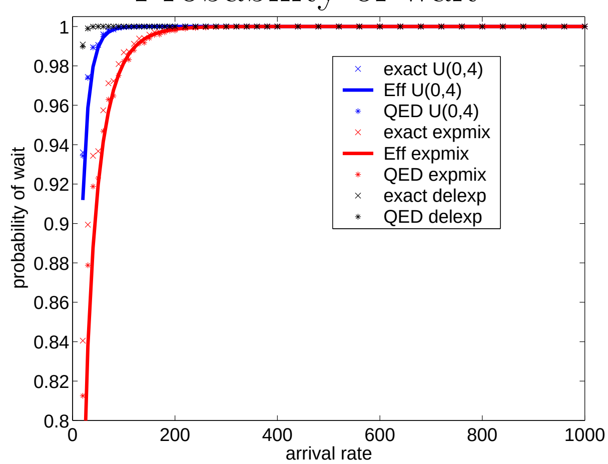
Average waiting time



Probability to abandon



Probability of wait



Fluid-limit ED approximations for P$\{$Ab$\}$ and E$[W]$ are better than QED.

Impact of Customers' Patience: Theoretical Results

Lemma. Consider M/M/n+G; λ, μ, and n fixed.
Assume that for two patience distributions G_1 and G_2:

$$\int_0^x \bar{G}_1(\eta)d\eta \;\geq\; \int_0^x \bar{G}_2(\eta)d\eta\,, \qquad x > 0\,.$$

Then,

a. $\mathrm{P}^1\{V > 0\} \;\geq\; \mathrm{P}^2\{V > 0\}$; $\mathrm{P}^1\{W > 0\} \;\geq\; \mathrm{P}^2\{W > 0\}$.

b. $\mathrm{P}^1\{\mathrm{Ab}\} \;\leq\; \mathrm{P}^2\{\mathrm{Ab}\}$; $\mathrm{P}^1\{\mathrm{Ab}|V > 0\} \;\leq\; \mathrm{P}^2\{\mathrm{Ab}|V > 0\}$.

Proof. Follows from Baccelli and Hebuterne.

Theorem 1. In addition, fix average patience $\bar{\tau}$.
Let G_d be the deterministic patience distribution. Then, in steady state:

a. G_d maximizes the probabilities of wait $\mathrm{P}\{W > 0\}$ and $\mathrm{P}\{V > 0\}$.
b. G_d minimizes the probabilities to abandon $\mathrm{P}\{\mathrm{Ab}\}$ and $\mathrm{P}\{\mathrm{Ab}|V > 0\}$.
c. G_d maximizes the average wait $\mathrm{E}[W]$.
d. G_d maximizes the average queue length $\mathrm{E}[Q]$.

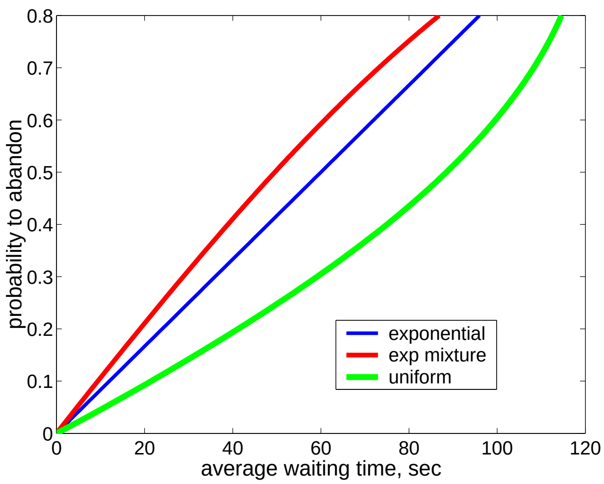
Proof. a+b. Follow from Lemma.
c. Functional maximization. Variation calculus.
d. Follows from Little's formula.

Theorem 2. For lightly-loaded M/M/n+G queue ($\lambda \to 0$), linear $\mathrm{P}\{\mathrm{Ab}\}/\mathrm{E}[W]$ relation established.

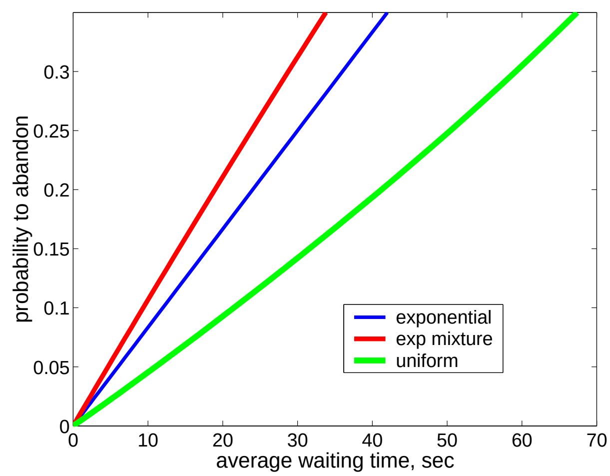
Impact of Customers' Patience: Numerical Results

Examples of linear P{Ab}/E[W] relations

Distributions: Exp(mean=2), Uniform(0,4), Hyperexponential.

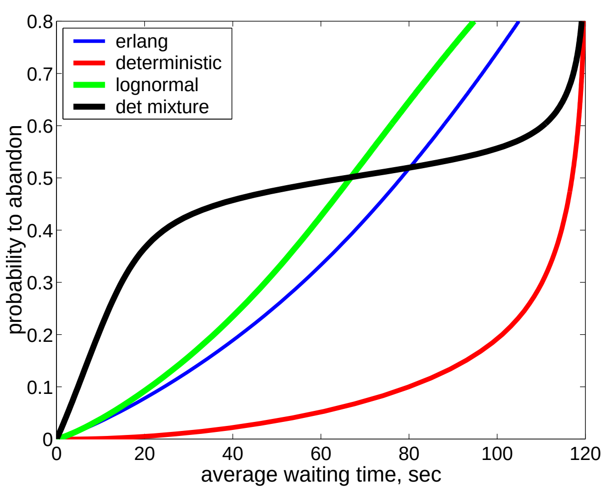


moderate loads

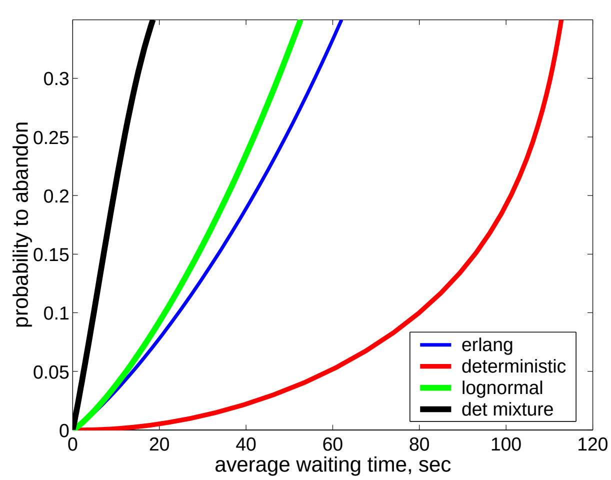


Examples of non-linear relations

Distributions: Deterministic(2), Erlang, Lognormal(2,2), mixture of two constants (0.2,3.8).



moderate loads



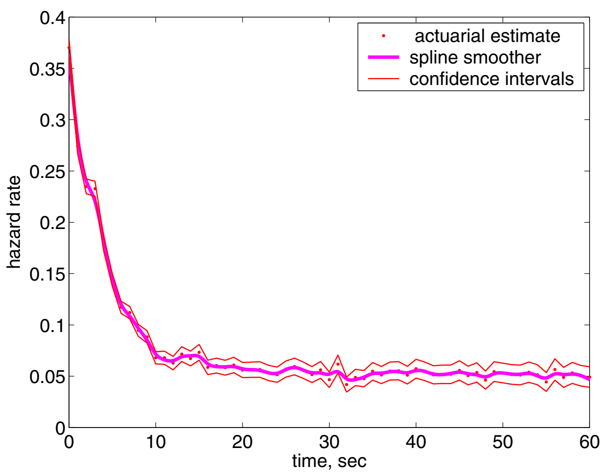
Some applications to call centers

Large US bank.
Daily volume 70,000 calls; 900-1200 agents positions on weekdays.
Two service types analyzed for 5 months.

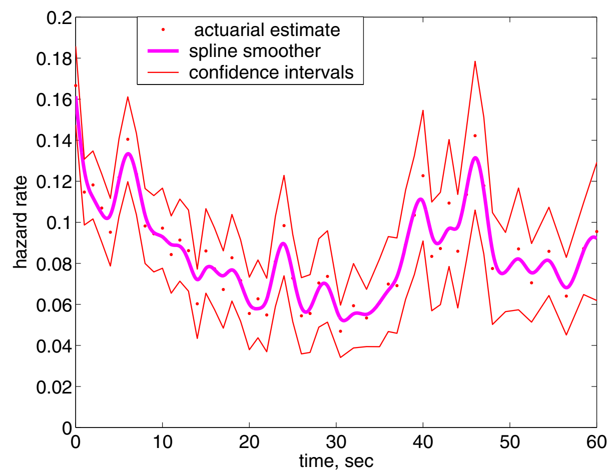
	Calls	E[S]	P$\{W > 0\}$	P{Ab}	E[W]
Retail	3,451,743	224.6 sec	30.6%	1.16%	6.33 sec
Telesales	349,371	453.9 sec	24.3%	1.76%	9.66 sec

Estimates of hazard rate

retail



telesales



Problems/Challenges:

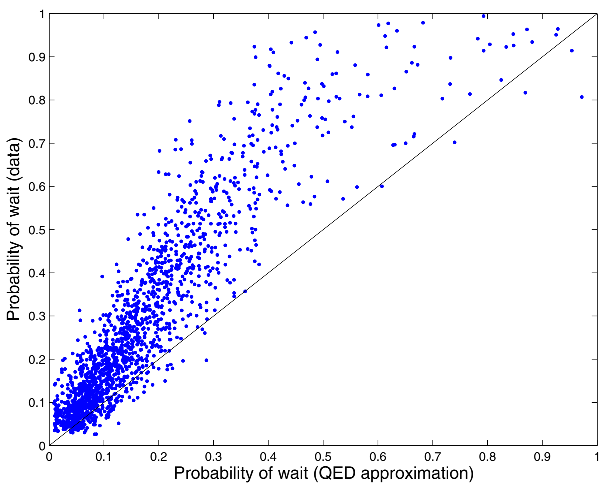
- Reliable data for number of agents n unavailable;
- Significant variability of hazard rate/density near the origin.

Approach: Estimate n via some performance measure (P{Ab}).
Fit other performance measure(s).

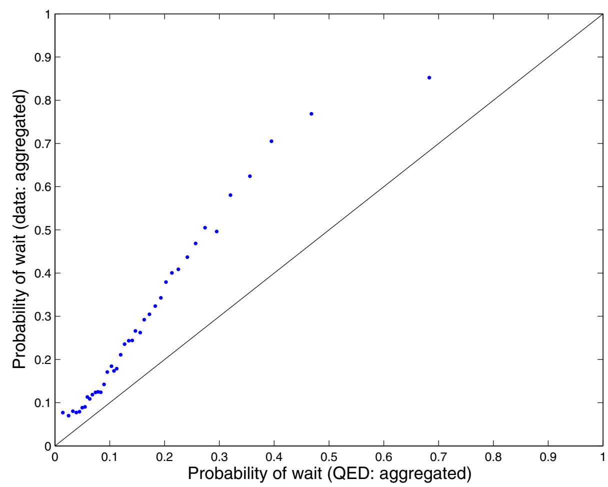
Substitute g_0 := estimate of hazard (density) at the origin.
We observe bad fit for the two examples below.

Retail: fitting probability of wait.

hourly data

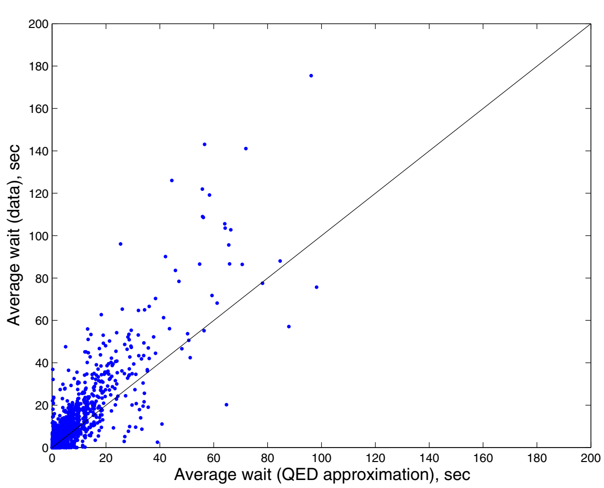


aggregated

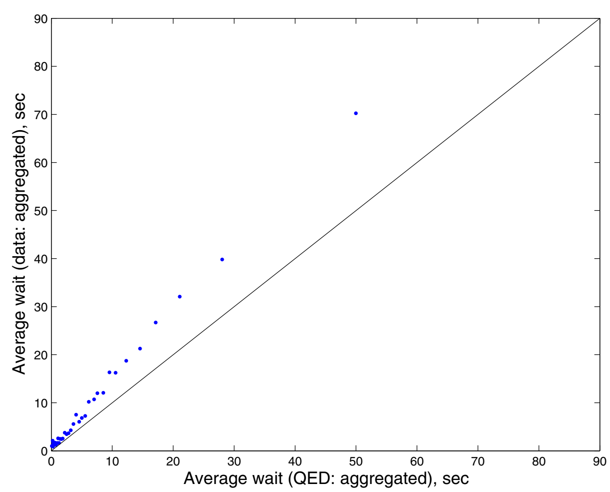


Telesales: fitting average wait.

hourly data

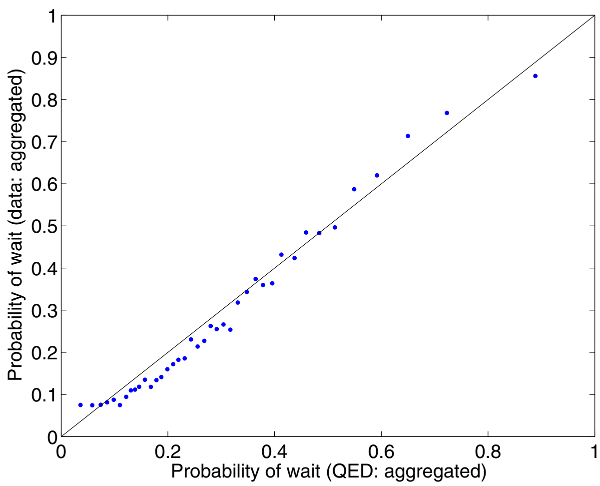


aggregated

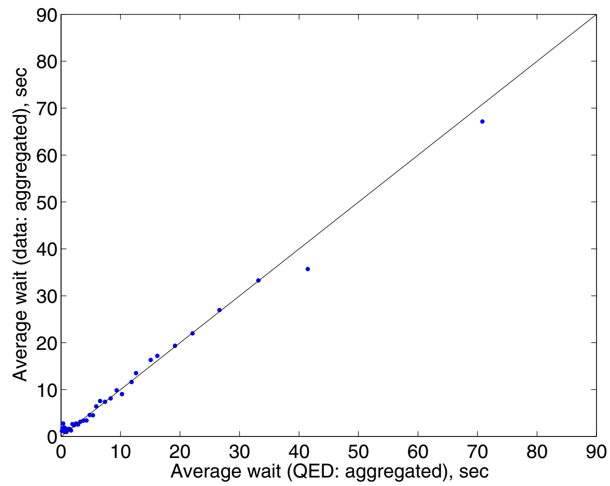


Solution: Substitute $g_0 :=$ overall $\mathrm{P}\{\mathrm{Ab}\}/\mathrm{E}[W]$ to QED formulae.

Retail. $\mathrm{P}\{W > 0\}$

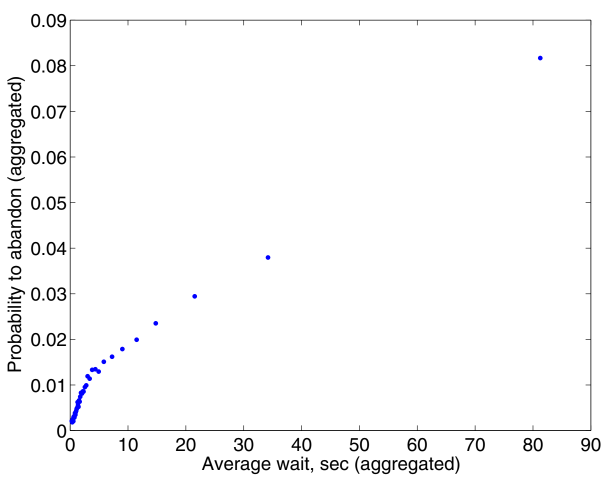


Telesales. $\mathrm{E}[W]$

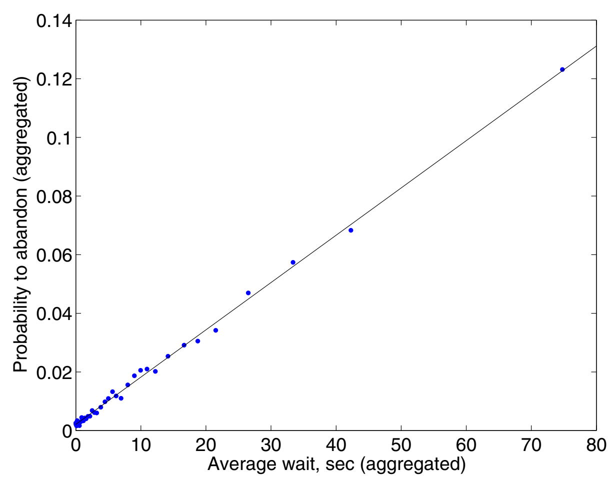


$\mathbf{P\{Ab\}/E[}\boldsymbol{W}\mathbf{]}$ relation

Retail



Telesales



For telesales, hazard variability near the origin much smaller. Hence, pattern much closer to straight line.

Conclusions

QED approximation: Can be performed using any software that provides the standard normal distribution (e.g. Excel). Works well for

- Number of servers n from 10's to 1000's;
- Agents highly utilized but not overloaded ($\sim$90-98%);
- Probability of delay 10-90%;
- Probability to abandon: 3-7% for small n, 1-4% for large n.

ED approximation: Requires solving equation $G(x) = \gamma$, and integration (calculating $H(x^*)$). Works well for

- Number of servers $n \geq 100$.
- Agents very highly utilized (close to 100%);
- Probability of delay: more than 85%;
- Probability to abandon: more than 5%.

QD approximation: preferable only for very high-performance systems.

Linear P{Ab}/E[W] relation: prevails in a broad context:

- QED and QD operational regime;
- Many non-exponential patience distributions (practically);
- Lightly loaded systems;
- Real data.

Possible Future Research

- **Dimensioning** M/M/n+G queue.
  Formal framework for *quality-efficiency tradeoff*.
  Minimize sum of staffing, waiting and abandonment costs.
  For Erlang-C, solved in Borst, Mandelbaum, Reiman (2004).
- Queues with random arrival rate.
- Queues with time-inhomogeneous arrival rate
  (ongoing work of Zohar Feldman).
- More data analysis.
- Generally distributed service times: M/G/n+G;
  (recent papers of Whitt (2004)).
- M/M/n+G: queue-length distribution;
- Process-limit results for M/M/n+G;