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Review: Markov Jump-Processes (MJP)

MJP $X = \{X_t,\ t \geq 0\}$ on $\mathcal{S} = \{i, j, \ldots\}$ countable.
Markov property: $P_r\{X_t = j | X_r,\ r < s;\ X_s = i\} = P_{ij}(s,t), \quad \forall\, s < t,\ \forall\, i,j \in \mathcal{S}$.
Time homogeneity: $P_r\{X_{s+t} = j | X_s = i\} = P_{ij}(t), \quad \forall\, s,t,\ i,j,$ transition probabilities.

Characterization: π^0 = initial distribution and $P(t) = [P_{ij}(t)],\ t \geq 0$, stochastic.
Finite-dimensional distributions:
$P_r\{X_0 = i_0,\ X_{t_1} = i_1, \ldots, X_{t_n} = i_n\} = \pi^0(i_0) P_{i_0,i_1}(t_1) \ldots P_{i_{n-1},i_n}(t_n - t_{n-1})$.

$P(t)$: stochastic ; $P(s+t) = P(s)P(t), \quad \forall\, s,t$ (Chapman Kolmogorov);
$\exists\, P(0) = I$; $\exists\, \dot{P}(0) = Q = [q_{ij}]$, infinitesimal generator $\left(\sum_{j \in \mathcal{S}} q_{ij} = 0\right)$.

Micro to Macro : $\dot{P}(t) = P(t)Q \ \ (= QP(t))$ and $P(0) = I$
Forward (Backward) equations.

Solution : $P(t) = \exp[tQ] = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n$, $t \geq 0$.

Animation: $i \xrightarrow{q_{ij}} j; \quad \forall\, i,j \in \mathcal{S}\ \exists$ exponential clock at rate q_{ij}, call it (i,j).
Given i, consider clocks $(i,j),\ j \in \mathcal{S}$; move to the "winner" when rings.
Thus: stay at $i \sim \exp(q_i = \sum_{j \neq i} q_{ij})$ and switch to j with probability $P_{ij} = q_{ij}/q_i$
$(q_{ij} = q_i P_{ij}, i \neq j; q_{ii} = -q_i)$.

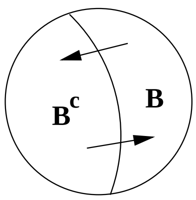
Transient analysis	vs. long-run/limit	stability/steady-state
	$\exists\, \lim_{t\uparrow\infty} P_{ij}(t) = \pi_j,\ \forall\, i;$	$\pi = \pi P(t),\ \forall\, t.$

Calculation via **steady-state equations:** $\dot{P}(\infty) = P(\infty)Q \Rightarrow \left\{ \begin{array}{l} 0 = \pi Q \\ \sum_i \pi_i = 1,\ \pi_i \geq 0 \end{array} \right\}$

or balance equations: $\sum_{i \neq j} \pi_i q_{ij} = -\pi_j q_{jj} = \sum_{i \neq j} \pi_j q_{ji},\ \forall\, j$.

Transition rates: $\pi_i q_{ij}$ = long-run average number of switches from i to j.

Cuts: $\sum_{i \in B} \sum_{j \in B^c} \pi_i q_{ij} = \sum_{i \in B^c} \sum_{j \in B} \pi_i q_{ij},\ \forall\, B \subset \mathcal{S}$.



Ergodic Theorem: Let X be *irreducible* ($i \leftrightarrow j$). Assume that there exists a solution π to its steady-state equations. Then, X must be "unexplosive" and π must be its stationary distribution, its limit distribution and

SLLN $\bullet \lim\limits_{T\uparrow\infty} \frac{1}{T} \int_0^T f(X_t)dt = \sum\limits_i \pi_i f(i)$ ("$=$"$Ef(X_\infty)$) ; eg. $f(x) = 1_B(x)$.

$\bullet \lim\limits_{T\uparrow\infty} \frac{1}{T} \sum\limits_{t\le T} g(X_{t-}, X_t) = \sum\limits_i \pi_i \sum\limits_j q_{ij} g(i,j)$, for $g(x,x) = 0$, $\forall\, x$; e.g. $g(x,y) = 1_C(x,y)$.

Birth-and-death process: MJP on $S = \{0,1,2,\ldots\}$, where all jumps are between adjacent states: $q_{ij} = 0$ if $|i-j| > 1$.

Cuts: $\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}$.
(Take $B = \{0,1,\ldots,i\}$ and $B^c = \{i+1, i+2, \ldots\}$.)

Reversibility: A stochastic process $X = \{X_t,\ -\infty < t < \infty\}$ is called *reversible* if for any τ

$$\{X_t,\ -\infty < t < \infty\} \overset{d}{=} \{X_{\tau - t},\ -\infty < t < \infty\}.$$

Fact. Ergodic MJP in the steady-state is reversible if and only if the *detailed balance equations* hold:

$$\pi_i q_{ij} = \pi_j q_{ji}, \quad \forall\, i,j \in \mathcal{S}.$$

Corollary. Every ergodic birth-and-death process is reversible.
(Follows from cut equations.)