

The Offered-Load in Fork-Join Networks

Applications to Staffing of Emergency Departments

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May 25, 2011

Introduction - The Offered Load

The Offered Load: The Stationary Case

hours of work (= service) that arrive per hour.

Example:

$\lambda = 20$ patients/hour; $E[S] = 0.5$ hours. Offered-Load $R = 20 \cdot 0.5 = 10$ hour of work per hour.

The Offered Load of an $M_t/GI/N_t$ queue

For the $M_t/GI/N_t$ queue, the *offered load* $R = \{R(t), t \geq 0\}$ is given by the function $R(t) = E[L(t)]$, where $L(t)$ is the number of customers/patients (=number of busy servers) at time t , in the corresponding $M_t/GI/\infty$ **queue**.

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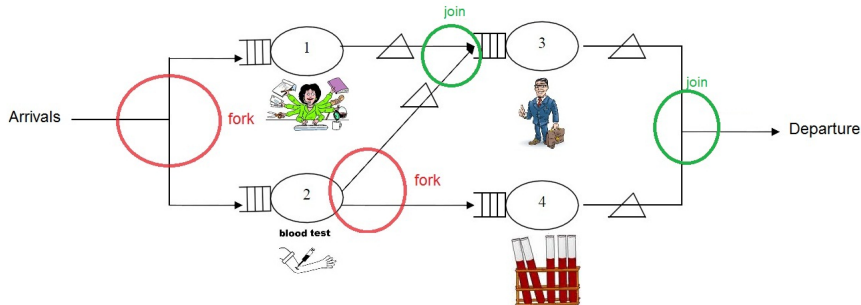
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- Offered-Load has been proved to be the skeleton for staffing of time-varying systems.

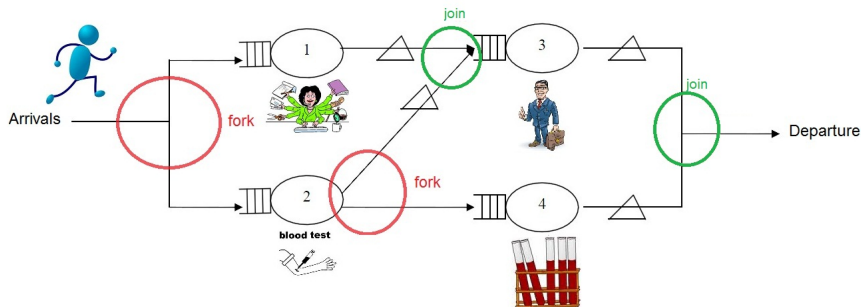
Fork-Join Networks

A fork-join network consists of service stations, which serve arriving customers simultaneously and sequentially according to pre-designed deterministic precedence constraints.



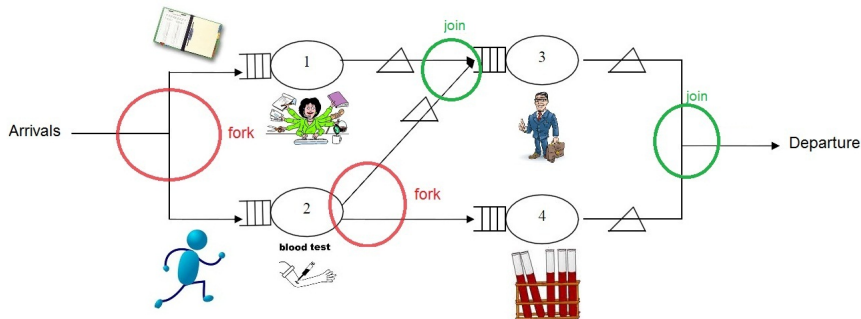
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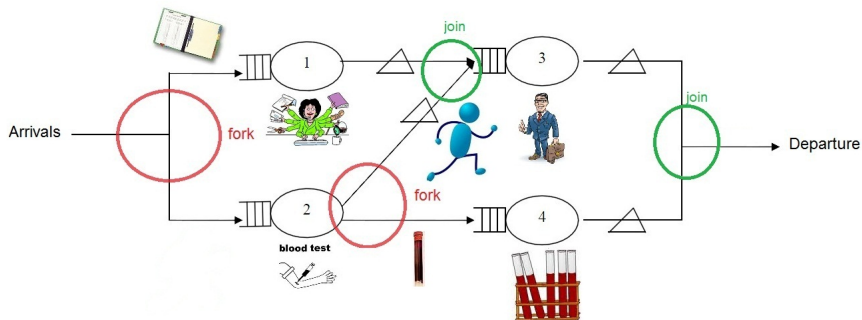
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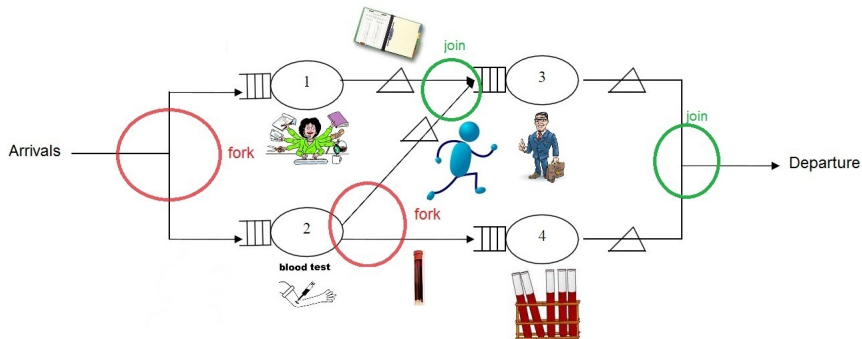
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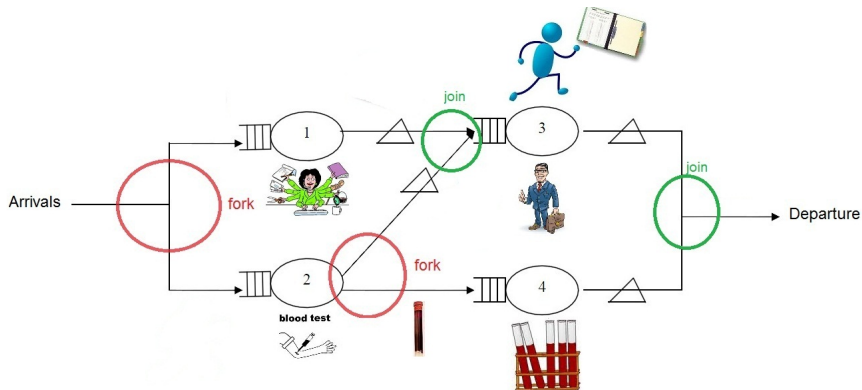
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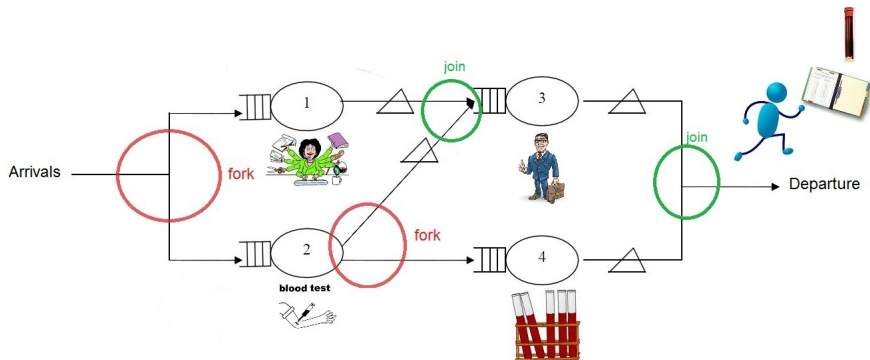
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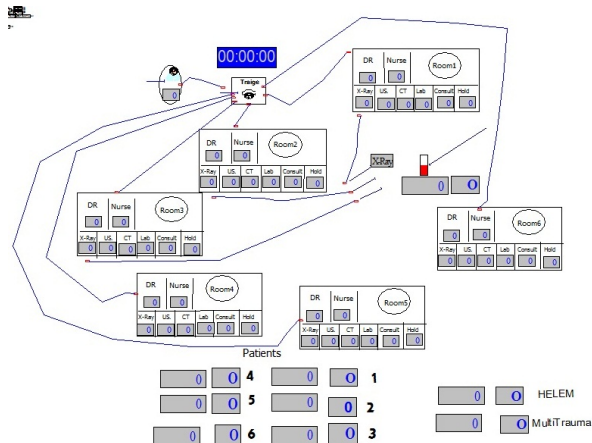
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Calculations and Applications using an ED simulator

In his MSc thesis, Yariv Marmur developed a generic ED simulation, using Rockwell's software "Arena".



Offered-Load calculation using the ED simulator:

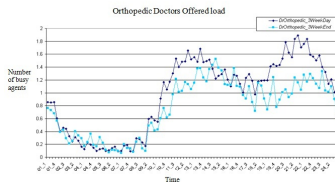
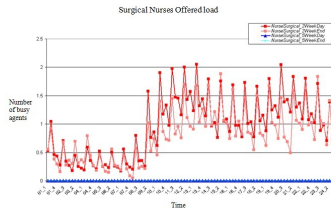
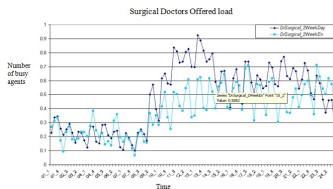
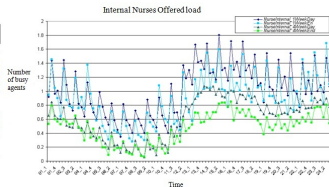
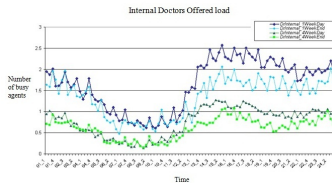
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- In analytical models, the Offered-Load is the number of patients in the corresponding infinite-server network.

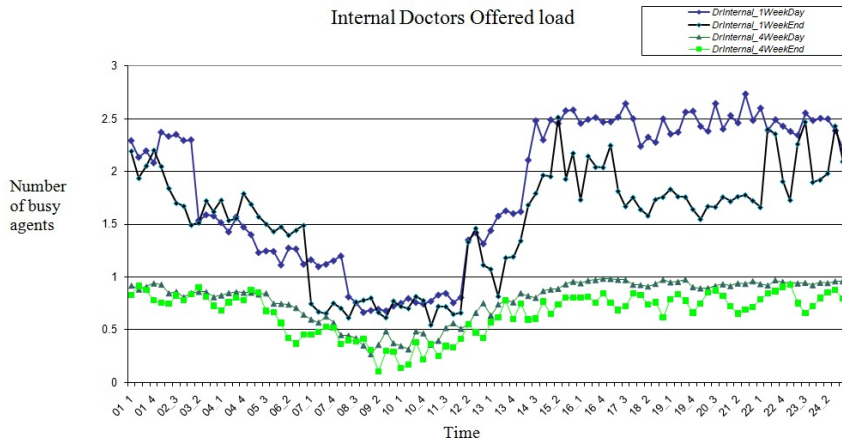
Offered-Load calculation using the ED simulator:

- In analytical models, the Offered-Load is the number of patients in the corresponding infinite-server network.
- We thus calculate the Offered-Load using the ED simulator, where we set the number of resources to be ∞ (at all the resources or only the resources under consideration).

The Offered-load of Resources: Doctors, Nurses



Zoom in on Internal Doctors



Using the square-root staffing rule:

$$N(t) = \text{round}(R(t) + \beta \cdot \sqrt{R(t)}) ,$$

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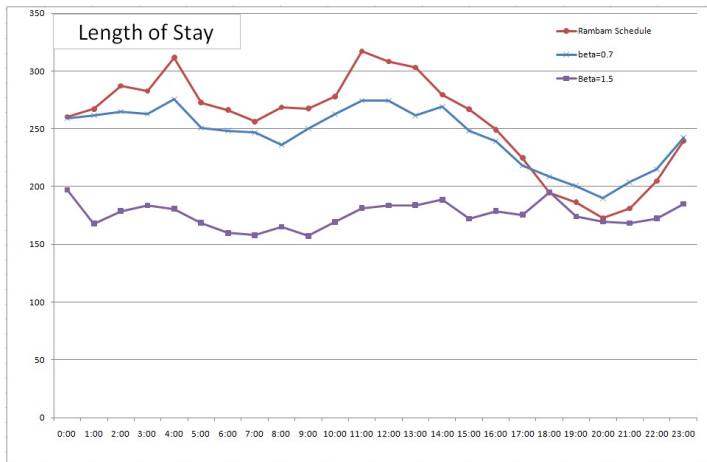
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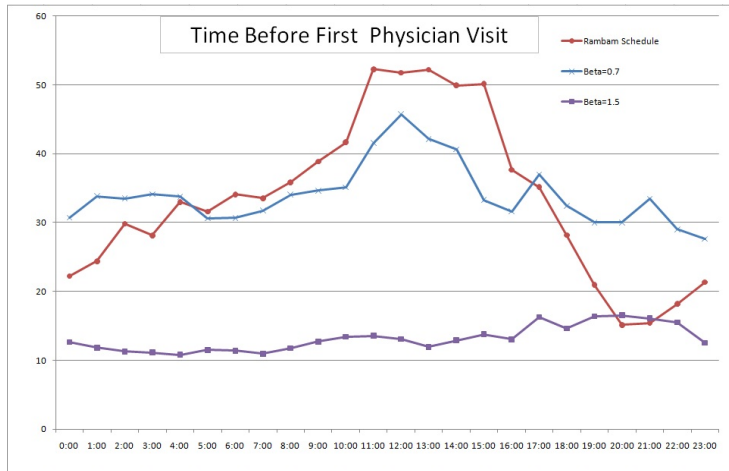
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- $N(t)$ are the number of agents at time t .
- β is "safety staffing".
- $\text{round}(x) = \begin{cases} \lceil x \rceil & \text{if } \lceil x \rceil - x \geq 0.5 \\ \lfloor x \rfloor & \text{if } \lceil x \rceil - x < 0.5 \end{cases}$.

Length of Stay

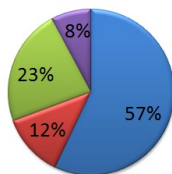


Time Up to First Physician Visit



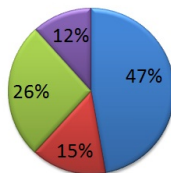
Different β 's vs Original Staffing

Anatomy of Sojourn Time



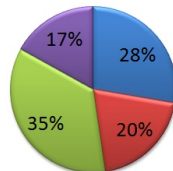
Current Staffing

LOS=7.1 hours



$\beta = 0.7$

LOS=6 hours



$\beta = 1.5$

LOS=4.6 hours

Daily Hours of Internal Dr: 56

63

86

Daily Hours of Internal Nurses: 48

41

68

■ Queue Time Waiting for a resource

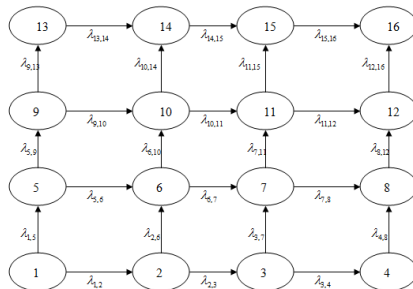
■ Service Time Duration of Service

■ Clinical Treatment Time Waiting for a medicine to take effect, waiting for an X-Ray to develop, etc.

■ Sync Time Waiting for a "Partner"

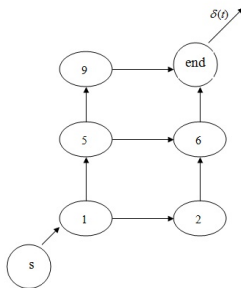
The Black Box Method

Consider a fork join network with V its set of nodes and A its set of arcs. To calculate the Offered Load of station i , one can define a new fork join network i^- , as follows:



The Black Box Method

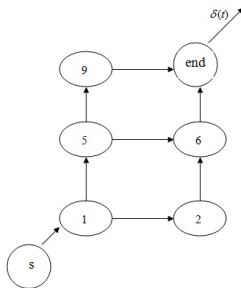
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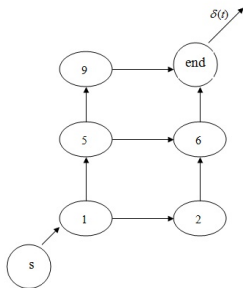
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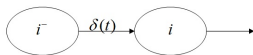
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- Let V^- be all the nodes of V which have a directed path to station i .
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- Let i^- be the fork-join network with V^- its set of nodes and A^- its set of arcs.



Example: the network 10^-

We now have a Tandem of 2 service stations, where queue i is the second queue.



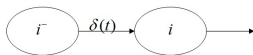
The Offered-Load at station i

The offered load at station i , is given by

$$R_i(t) = E(\lambda(t - T_i - S_i^e)) \cdot E(S_i), \quad t \geq 0,$$

where T_i is the total sojourn time of network i^- and S_i^e is the residual service time at station i .

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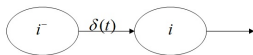
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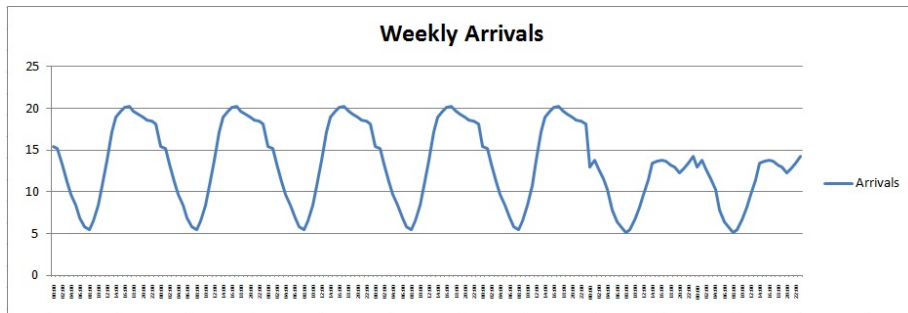
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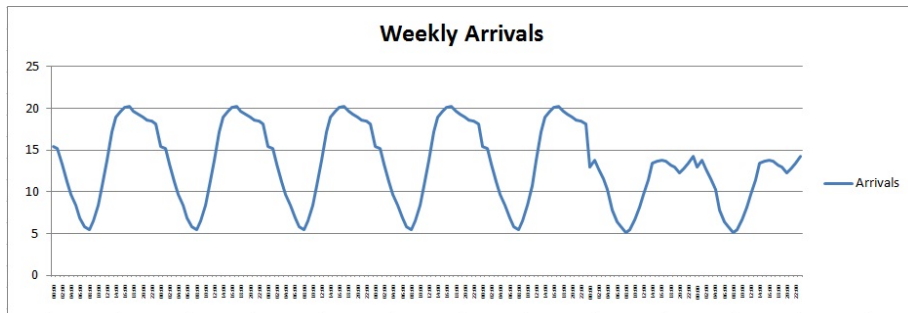
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- Thus, finding the distribution of T_i is required for calculating the Offered-Load of station i .
- S_i^e 's density function is $f_{S_i^e}(x) = \frac{1 - F_{S_i}(x)}{E(S_i)}$.

Periodic Arrival Rate: Actual Arrival Rate

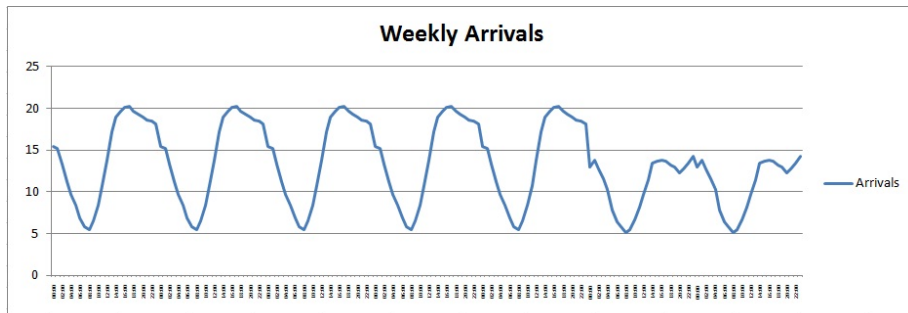


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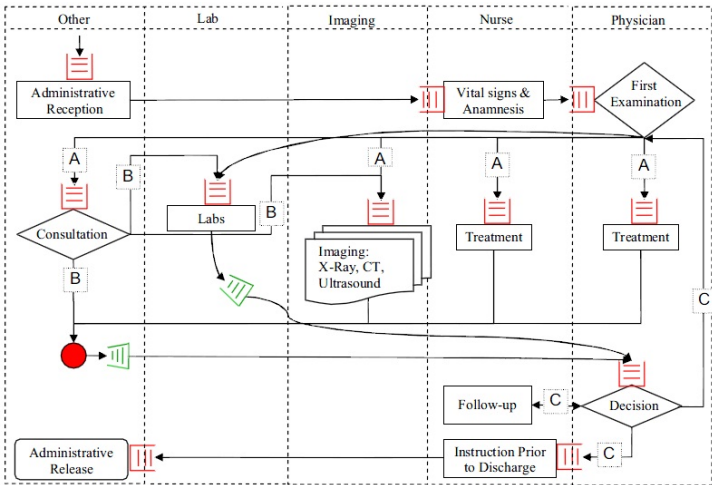
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- The example of calculations were applied to the following fork-join network:

Periodic Arrival Rate: Actual Arrival Rate



Example: The Offered-Load of Internal Drs:

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- ② We then calculate the Offered-Load of each one of the activities.
- ③ Summing up the Offered-Load of the activities will give us the Offered-Load of Internal Dr's.

- **Example: The Offered-Load of Internal Drs:**

$$R(t) = \sum_{j=1}^3 R_{a_j}(t) = \sum_{j=1}^3 \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} E(e^{-\frac{2\pi i}{N} kS}) \cdot E(S_{a_j}),$$

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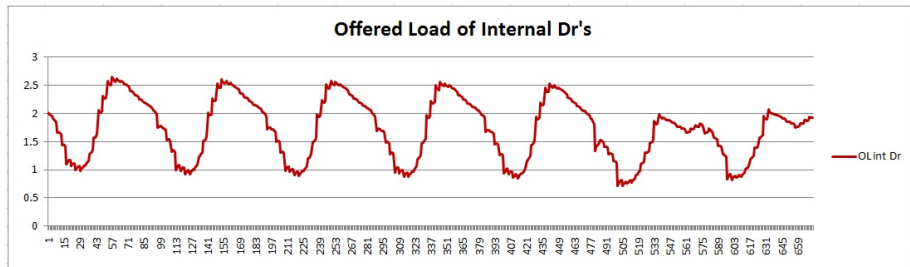
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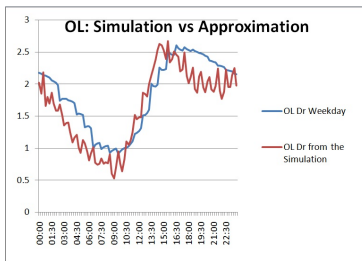
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- To calculate the expression $E(e^{-\frac{2\pi i}{N} kS})$ we disaggregated the complex ED network into a few simpler subnetworks.

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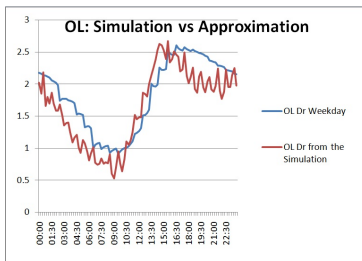
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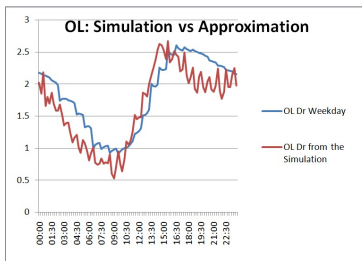
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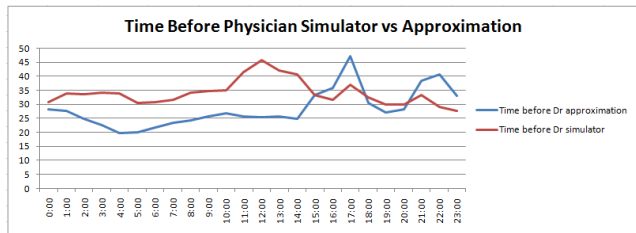
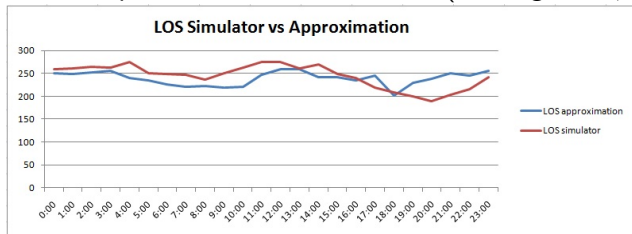
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- The method provides an Offered-Load approximation. Its advantages are:
 - 1 Easy to calculate, once a calculation algorithm has been established.
 - 2 One can use this method on other fork-join networks (other ED networks in particular); instead of programming a simulator.

Comparing results: Simulator vs Approximation

For example, results of internal Drs (staffing with $\beta = 0.7$):



Project Completion Time by Continues Time Markov Chain

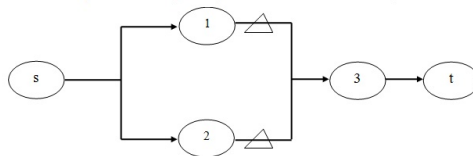
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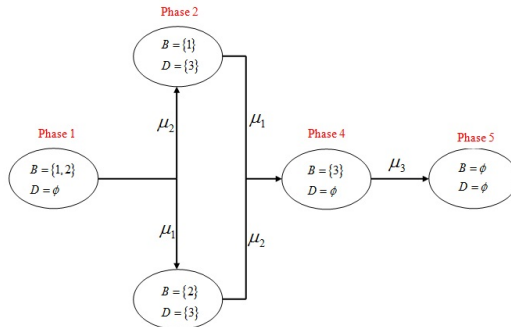
- We adopt a result by Adlakha and Kulkarni [1986], to represent a fork-join network as a continues time Markov chain (activities on arcs vs activities on nodes, in our case).
- The continues time Markov chain state space consists of all the pairs (B, D) : B denote all the active nodes and D all the dormant nodes.

Project Completion Time: Example

The fork-join Network: (here, μ_i is the service rate of station i)

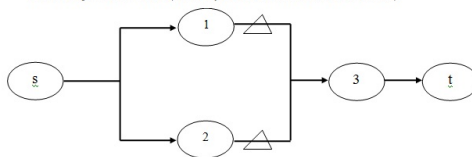


The CTMC rate diagram:

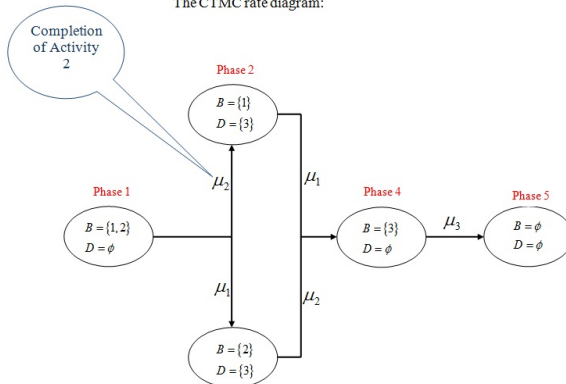


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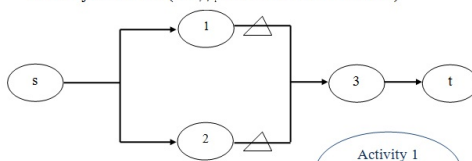


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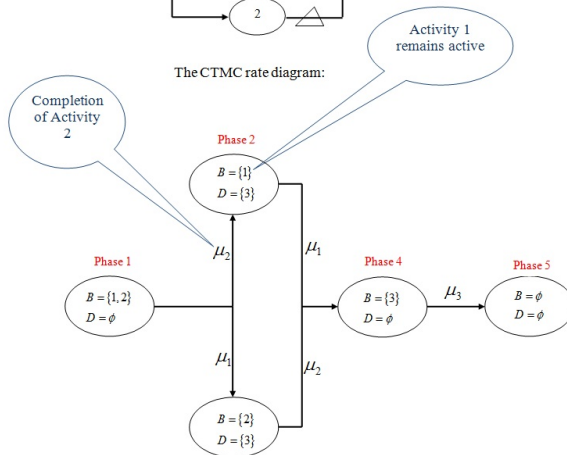


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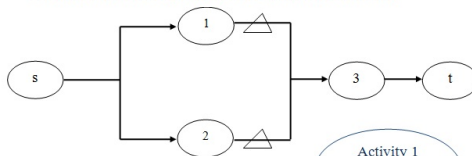


The CTMC rate diagram:

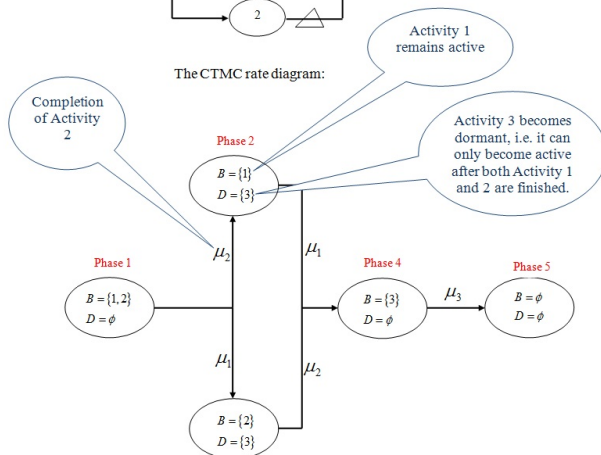


Project Completion Time: Example

The fork-join Network: (here, μ_i is the service time of station i)



The CTMC rate diagram:



Project Completion Time by Continuous Time Markov Chain

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Backward Algorithm

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$$p_i(0) = \delta_{iN}, \quad 0 \leq i \leq N,$$

where $\delta_{ij} = 1$ if $i=j$, and 0 otherwise, and q_{ij} are taken from the infinitesimal generator matrix of the continues time Markov chain.

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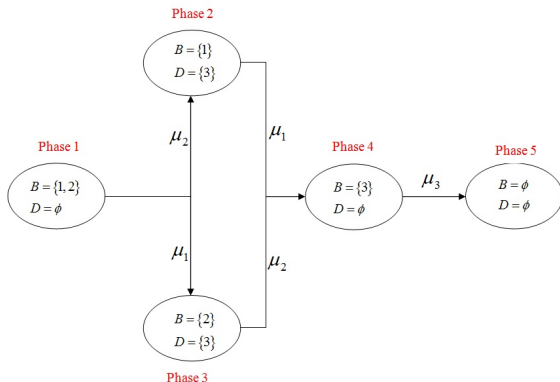
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- We start the algorithm with $p_N(t) \equiv 1$, for $t \geq 0$ and compute $p_{N-1}(t), \dots, p_1(t), p_0(t)$ recursively backward.

Project Completion Time by Continuous Time Markov Chain: Example



- $P_5(t) = 1,$
- $P'_3(t) = \mu_2 \cdot P_4(t),$
- $P'_1(t) = \mu_1 \cdot P_3(t) + \mu_2 \cdot P_2(t).$

- $P'_4(t) = \mu_3 \cdot P_5(t),$
- $P'_2(t) = \mu_1 \cdot P_4(t),$

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- Staffing according to the Offered Load stabilizes measures of performance of the emergency department over time.
- Tractable analysis of a complex network (fork-join) captures the full complexity of an emergency department.