The Offered-Load in Fork-Join Networks

Applications to Staffing of Emergency Departments

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Jointly with H.Kaspi and A.Mandelbaum

June 24, 2012

Introduction - The Offered Load

The Offered Load: The Stationary Case

hours of work (= service) that arrive per hour.

Example:
$\lambda = 20$ patients/hour; $E[S] = 0.5 hours$. Offered-Load $R = 20 \cdot 0.5 = 10$ hour of work per hour.

The Offered Load of an $M_t/GI/N_t$ queue

For the $M_t/GI/N_t$ queue, the *offered load* $R = \{R(t),\ t \geq 0\}$ is given by the function $R(t) = E[L(t)]$, where $L(t)$ is the number of customers/patients (=number of busy servers) at time t, in the corresponding $M_t/GI/\infty$ **queue.**

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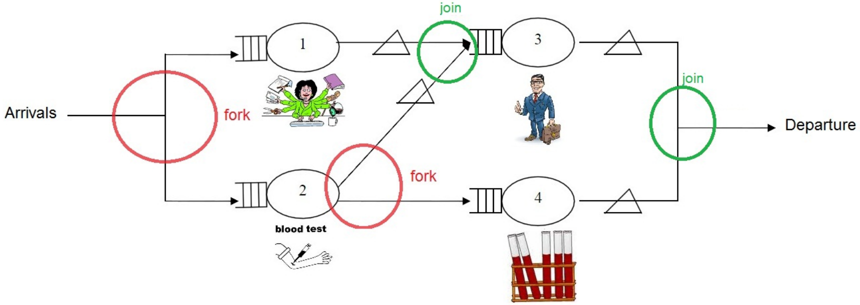
- Offered-Load has been proved to be the skeleton for staffing of time-varying systems.

Fork-Join Networks

A fork-join network consists of service stations, which serve arriving customers simultaneously and sequentially according to pre-designed deterministic precedence constraints.



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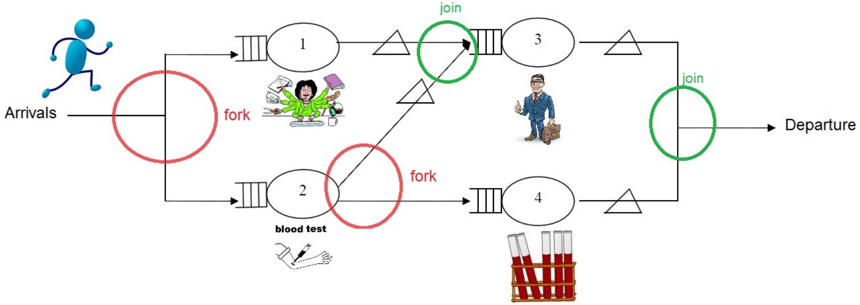

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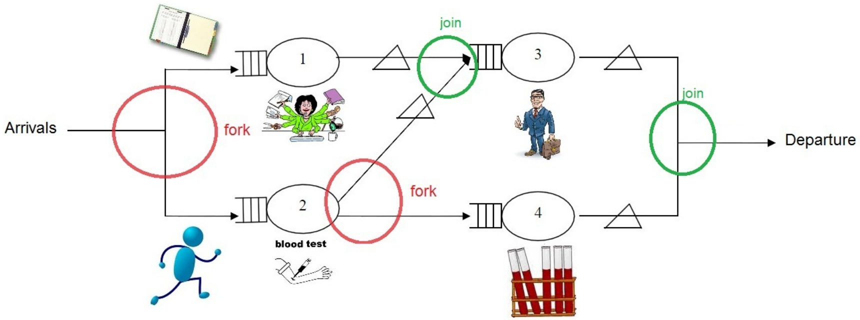

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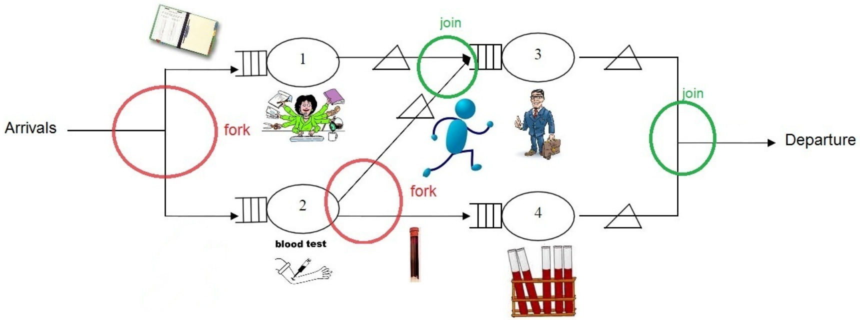

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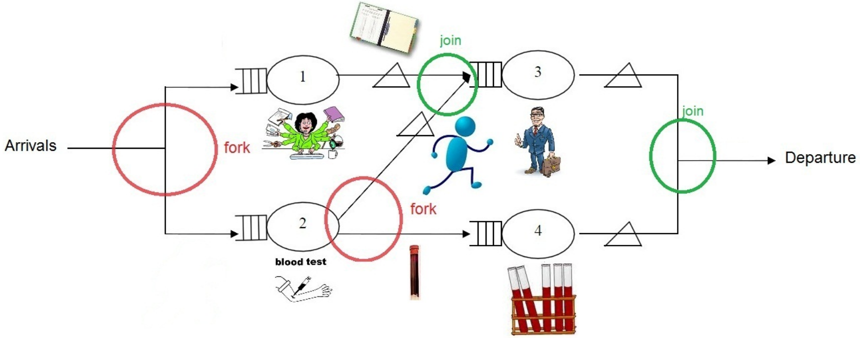

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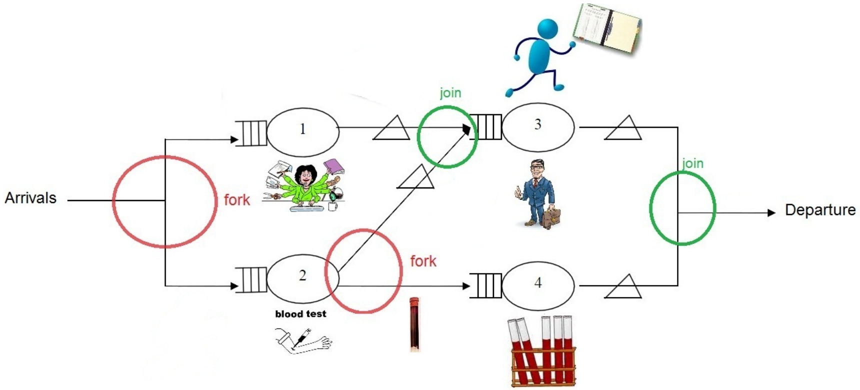

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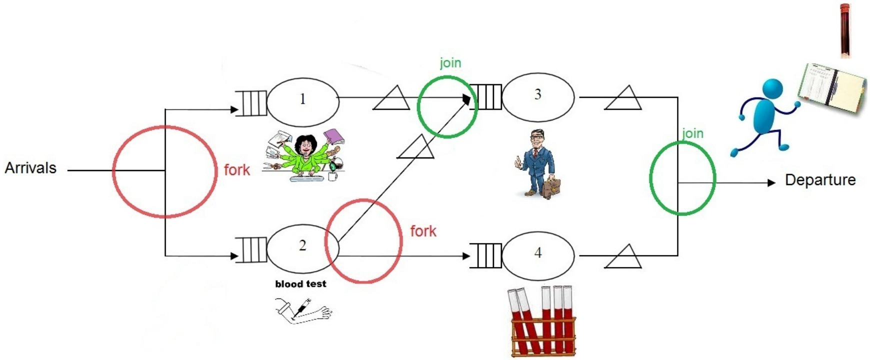
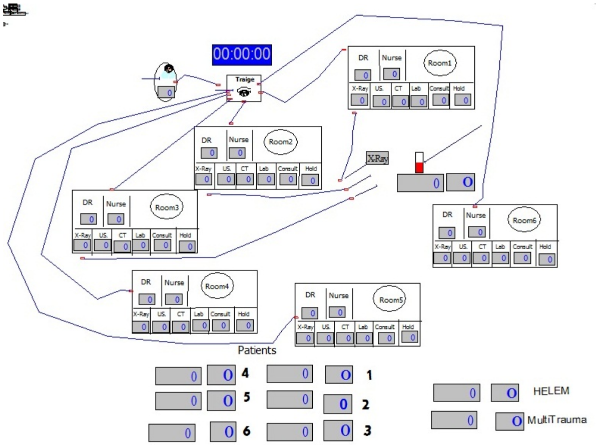


In his MSc thesis, Yariv Marmur developed a generic ED simulation, using Rockwell's software "Arena".



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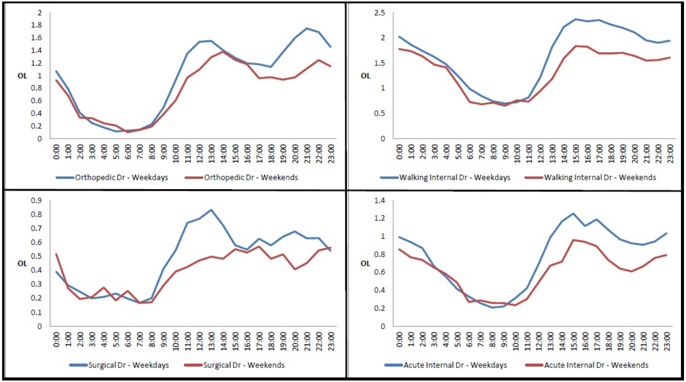
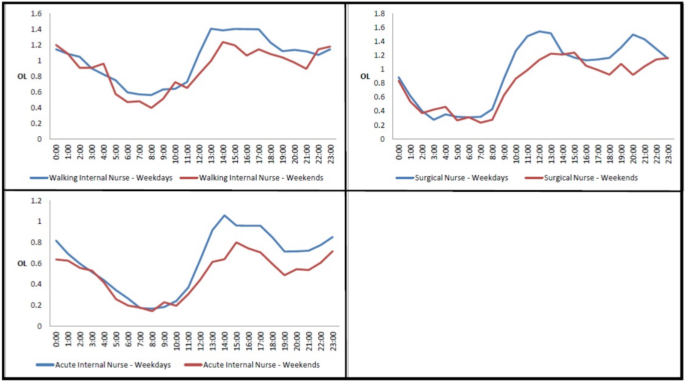
- We thus calculate the Offered-Load using the ED simulator, where we set the number of resources to be ∞ (at all the resources or only the resources under consideration).

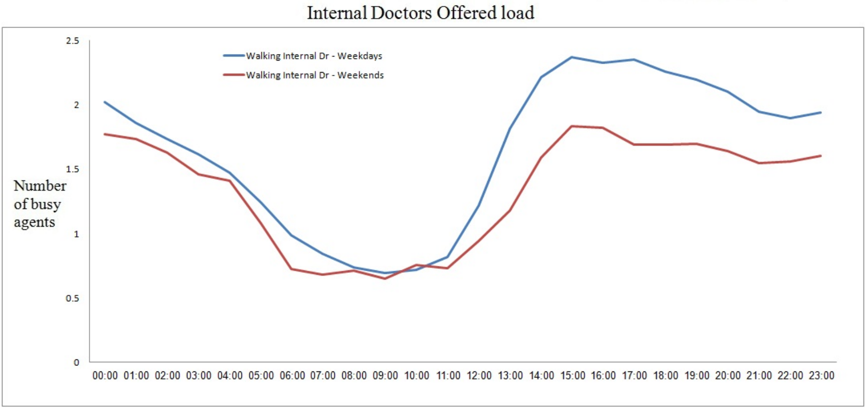
The Offered-Load of Resources: Doctors



The Offered-Load of Resources: Nurses



Zoom in on Internal Doctors



Staffing

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where:

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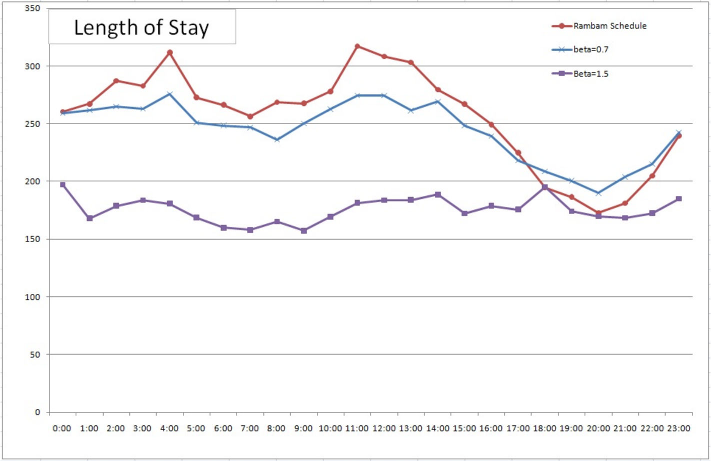
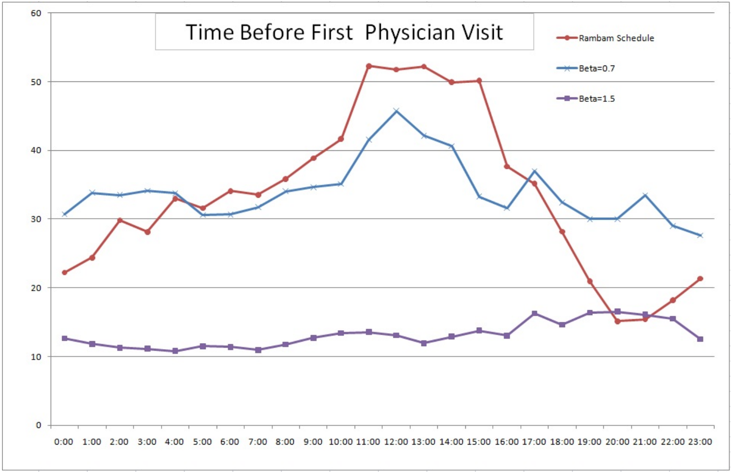
- β is "safety staffing".
- $round(x) = \left\{ \begin{smallmatrix} \lceil x \rceil & if & \lceil x \rceil - x \geq 0.5 \\ \lfloor x \rfloor & if & \lceil x \rceil - x < 0.5 \end{smallmatrix} \right.$.

Length of Stay

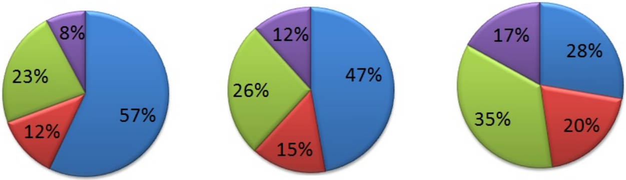


Time Up to First Physician Visit



Different β's vs Original Staffing

Anatomy of Sojourn Time

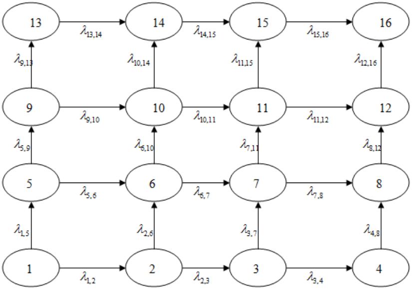


	Current Staffing	$\beta = 0.7$	$\beta = 1.5$
	LOS=7.1 hours	LOS=6 hours	LOS=4.6 hours
Daily Hours of Internal Dr:	56	63	86
Daily Hours of Internal Nurses:	48	41	68

- Queue Time — Waiting for a resource
- Service Time — Duration of Service
- Clinical Treatment Time — Waiting for a medicine to take effect, waiting for an X-Ray to develop,etc.
- Sync Time — Waiting for a "Partner"

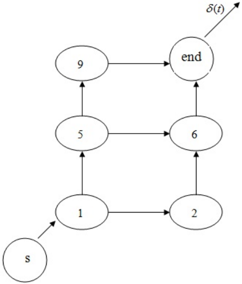
The Black Box Method

Consider a fork join network with V its set of nodes and A its set of arcs. To calculate the Offered Load of station i, one can define a new fork join network i^-, as follows:



The Black Box Method

- Let V^- be all the nodes of V which have a directed path to station i.



Example: the network 10^-

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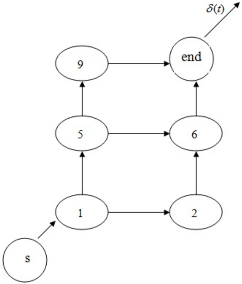

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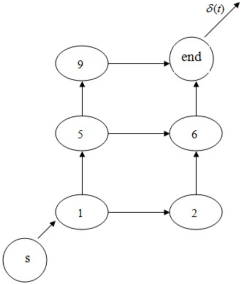
- Let i^- be the fork-join network with V^- its set of nodes and A^- its set of arcs.



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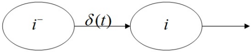
We now have a Tandem of 2 service stations, where queue i is the second queue.



The Offered-Load at station i

The offered load at station i, is given by

$$R_i(t) = E(\lambda(t - T_i - S_i^e)) \cdot E(S_i), \quad t \geq 0,$$

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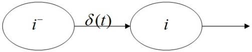

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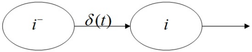

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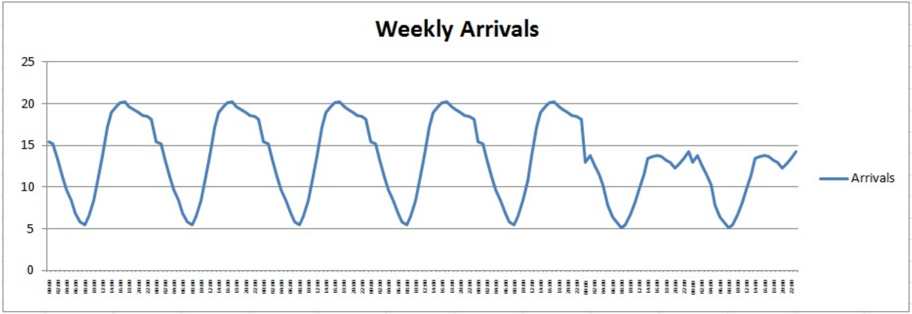
- Thus, finding the distribution of T_i is required for calculating the Offered-Load of station i.
- S_i^e's density function is $f_{S_i^e}(x) = \frac{1-F_{S_i(x)}}{E(S_i)}$.

Periodic Arrival Rate: Actual Arrival Rate



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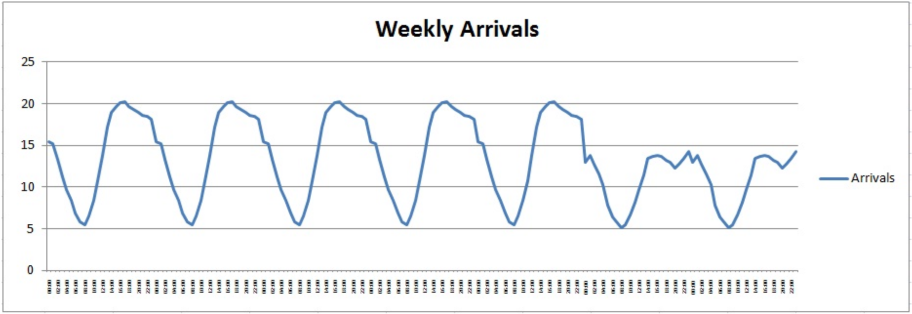

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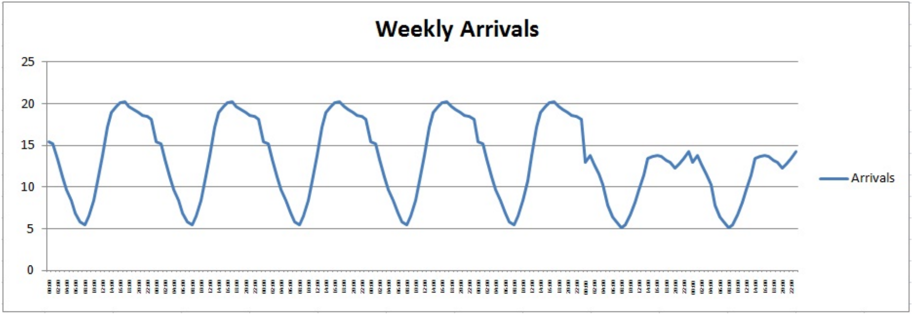

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- The example of calculations were applied to the following fork-join network:

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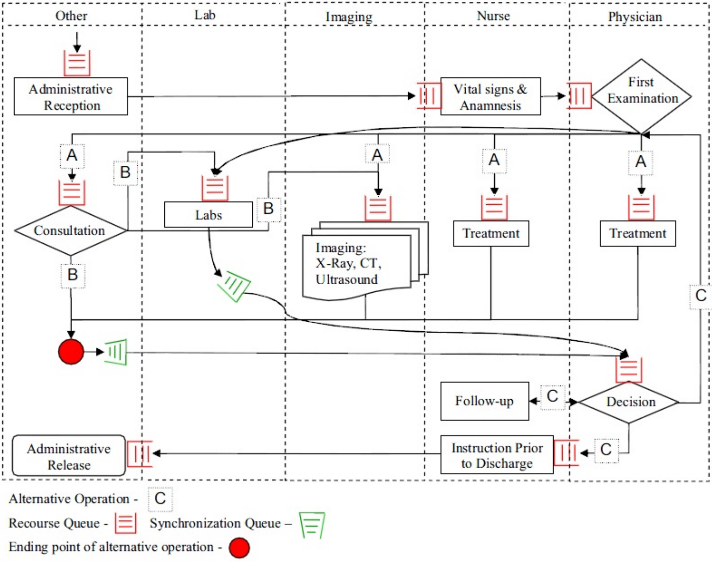

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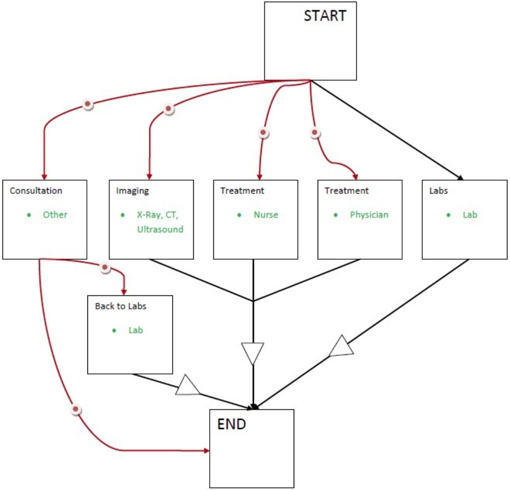
To calculate the Offered Load, we disaggregate the complex queueing network into simpler smaller networks:



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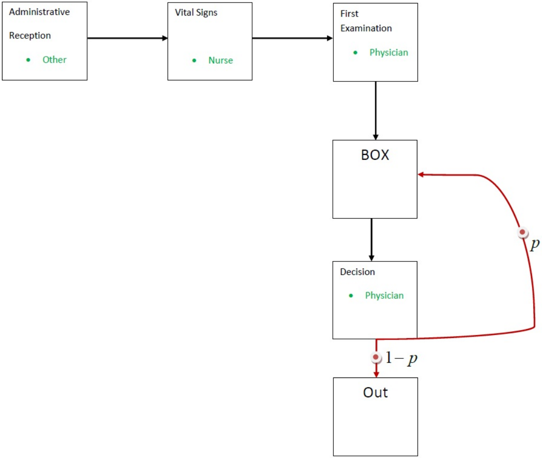
Looking at the whole network as one station, will yield the following simple open network.



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2. We then calculate the Offered-Load of each one of the activities.
3. Summing up the Offered-Load of the activities will give us the Offered-Load of Internal Dr's.

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- **Example: The Offered-Load of Internal Drs:**

$$R(t) = \sum_{j=1}^{3} R_{a_j}(t) = \sum_{j=1}^{3} \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} E(e^{-\frac{2\pi i}{N} kS}) \cdot E(S_{a_j}),$$

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- To calculate the expression $E(e^{-\frac{2\pi i}{N} kS})$ we disaggregated the complex ED network into a few simpler subnetworks.

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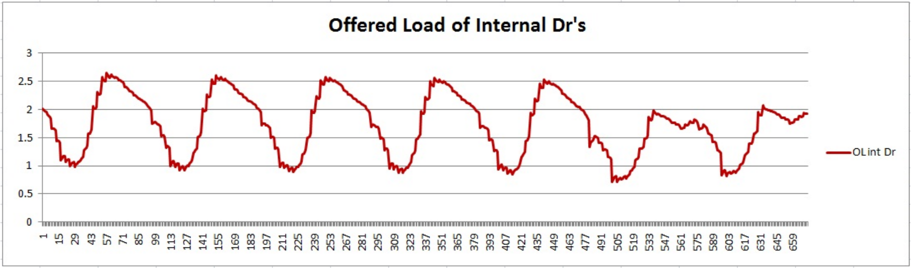

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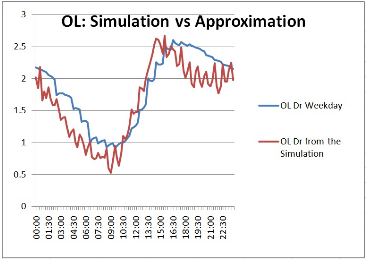
Comparison with the Offered-Load, calculated using the simulator



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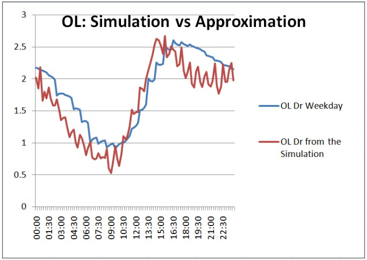

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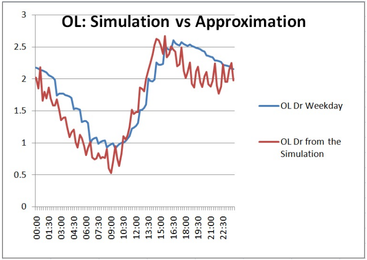

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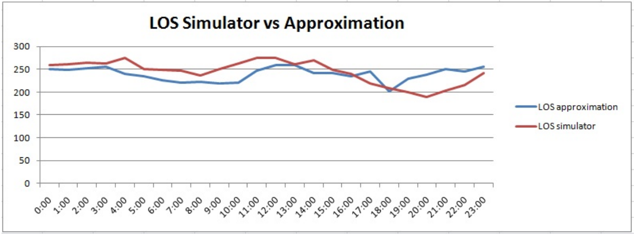
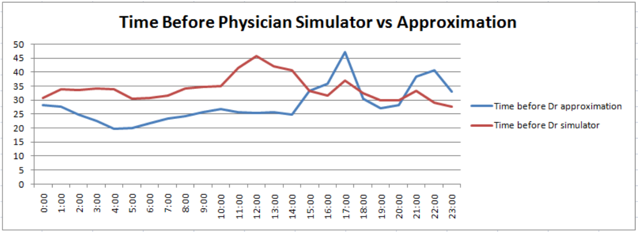
  1. Easy to calculate, once a calculation algorithm has been established.
  2. One can use this method on other fork-join networks (other ED networks in particular); instead of programming a simulator.

Comparing results: Simulator vs Approximation

For example, results of internal Drs (staffing with $\beta = 0.7$):





Project Completion Time by Continues Time Markov Chain

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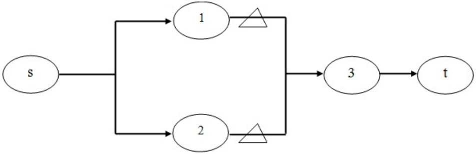
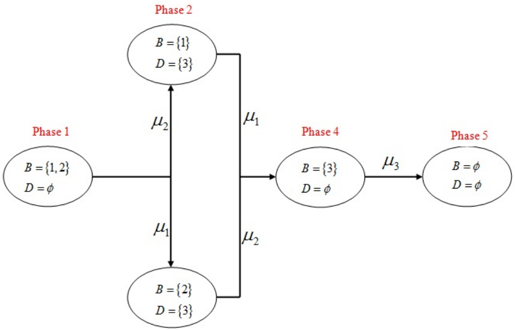
- The continues time Markov chain state space consists of all the pairs (B, D) : B denote all the <u>active</u> nodes and D all the <u>dormant</u> nodes.

Project Completion Time: Example

The fork-join Network: (here, μ_i is the service rate of station i)



The CTMC rate diagram:



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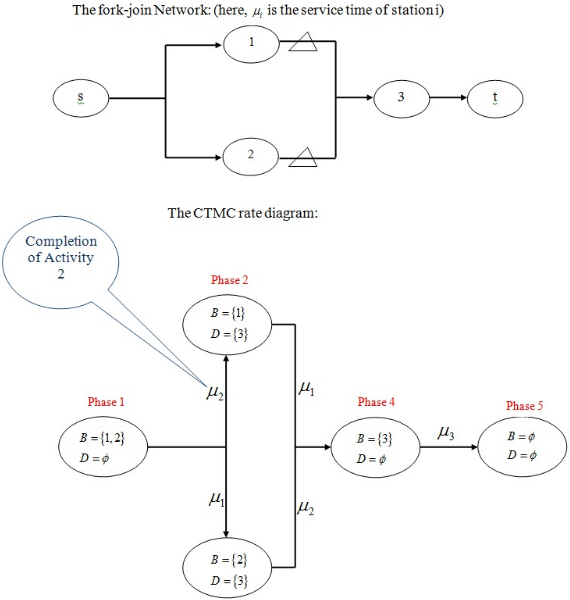

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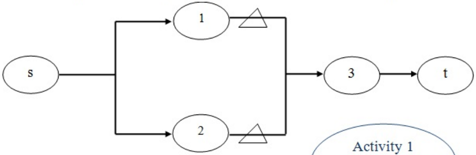

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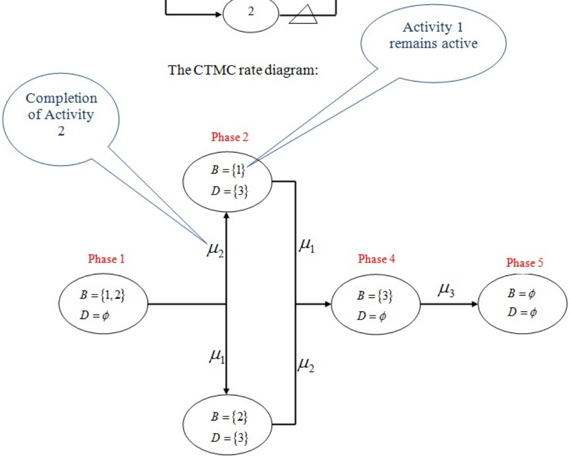

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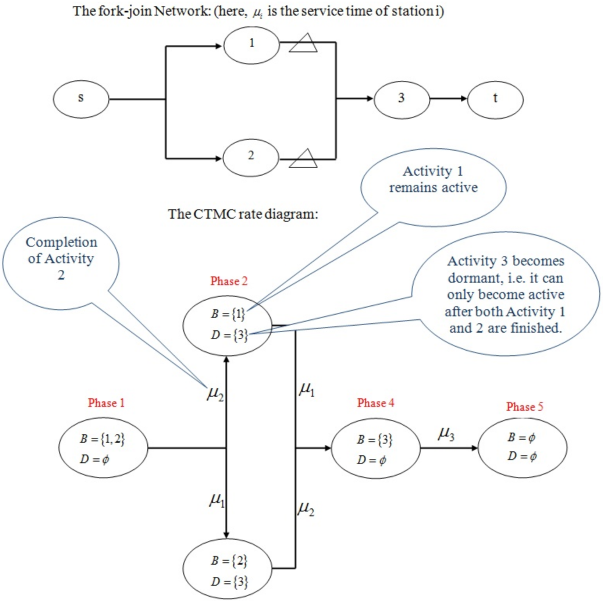

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where $\delta_{ij} = 1$ if i=j, and 0 otherwise, and q_{ij} are taken from the infinitesimal generator matrix of the continues time Markov chain.

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     where $\delta_{ij} = 1$ if i=j, and 0 otherwise, and q_{ij} are taken from the infinitesimal generator matrix of the continues time Markov chain.
- We start the algorithm with $p_N(t) \equiv 1$, for $t \geq 0$ and compute $p_{N-1}(t), ..., p_1(t), p_0(t)$ recursively backward.

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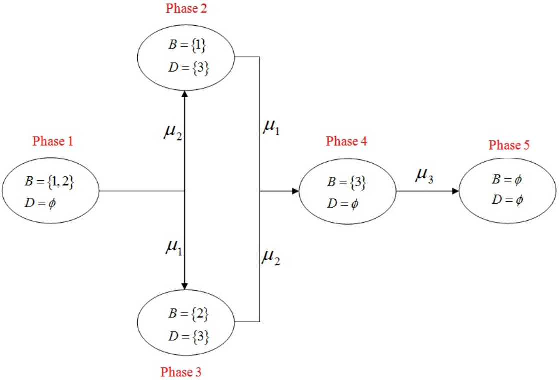

- $P_5(t) = 1$,
- $P_3'(t) = \mu_2 \cdot P_4(t)$,
- $P_1'(t) = \mu_1 \cdot P_3(t) + \mu_2 \cdot P_2(t)$.
- $P_4'(t) = \mu_3 \cdot P_5(t)$,
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- Tractable analysis of a complex network (fork-join) captures the full complexity of an emergency department.

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Questions
Thank You