The Offered-Load in Fork-Join Networks

Applications to Staffing of Emergency Departments

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July 18, 2012

The Offered Load: The Stationary Case

hours of work (= service) that arrive per hour.

Example:
$\lambda = 20$ patients/hour; $E[S] = 0.5 hours$. Offered-Load $R = 20 \cdot 0.5 = 10$ hour of work per hour.

The Offered Load of an $M_t/GI/N_t$ queue

For the $M_t/GI/N_t$ queue, the *offered load* $R = \{R(t),\ t \geq 0\}$ is given by the function $R(t) = E[L(t)]$, where $L(t)$ is the number of customers/patients (=number of busy servers) at time t, in the corresponding $M_t/GI/\infty$ **queue.**

Introduction - The Offered Load

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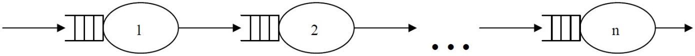
- Offered-Load has been proved to be the skeleton for staffing of time-varying systems.

Offered Load Calculations

The Offered Load in $M_t/GI/\infty$ queue - Representations (S.G. Eick, W.A. Massey, and W. Whitt)

For each t, $L(t)$ has a Poisson distribution with mean
$R(t) = E[L(t)] =$
$E[\lambda(t - S_e)] \cdot E[S] = E\left[\int_{t-S}^{t} \lambda(u)du\right] = \int_{-\infty}^{t} [1 - G(t-u)]\lambda(u)du,$
where
S is a generic service time with cdf G;
S_e is a generic "excess service time", having the following cdf:
$P(S_e \leq t) = \frac{1}{E(S)} \int_0^t [1 - G(u)]du, \qquad t \geq 0.$

Offered Load in Tandem Queues



The Offered Load at time t of the second station is given by

$$R_2(t) = E(\lambda(t - S_1 - S_{2e})) \cdot E(S_2), \quad t \geq 0.$$

Where S_{2e} is the excess service time.

The Offered Load of station number k ($k \leq n$) is:
$R_k(t) = E(\lambda(t - (S_1 + ... + S_{k-1}) - S_{ke})) \cdot E(S_k)$.

Periodic Arrival Rate: Example

Example: a tandem of two queues with Arrival rate
$\lambda(t) = 4 \cdot sin(1.5t) + 5 \quad t > -\infty$ and service times $S_i \sim exp(1)$, $i = 1, 2$.

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- $R_1(t) = E(\lambda(t - S_{1e})) \cdot E(S_1) = E(4 \cdot sin(1.5(t - S_{1e})) + 5) \cdot E(S_1) =$
  $E(4 \cdot \frac{e^{1.5i(t-S_{1e})} - e^{-1.5i(t-S_{1e})}}{2i} + 5) \cdot E(S_1) =$
  $\frac{4}{2i} \cdot E(e^{1.5i(t-S_{1e})}) \cdot E(S_1) - \frac{4}{2i} \cdot E(e^{-i1.5(t-S_{1e})}) \cdot E(S_1) + 5 =$
  $\frac{4}{2i} \cdot e^{1.5ti} \cdot \psi_{S1e}(1.5) \cdot E(S_1) - \frac{4}{2i} \cdot e^{-1.5ti} \cdot \psi_{S1e}(-1.5) \cdot E(S_1) =$
  $\frac{4}{2i} \cdot [\frac{e^{1.5it}}{1+1.5i}] - \frac{4}{2i} \cdot [\frac{e^{-1.5it}}{1-1.5i}] + 5$
  $= 1.25sin(1.5t) - 1.875 \cdot cos(1.5t) + 5$

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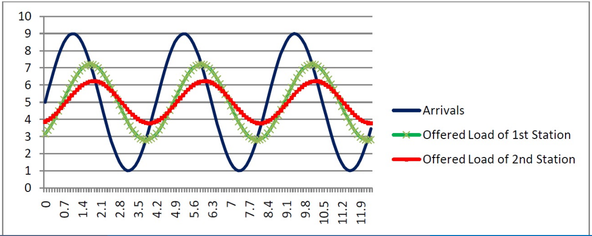
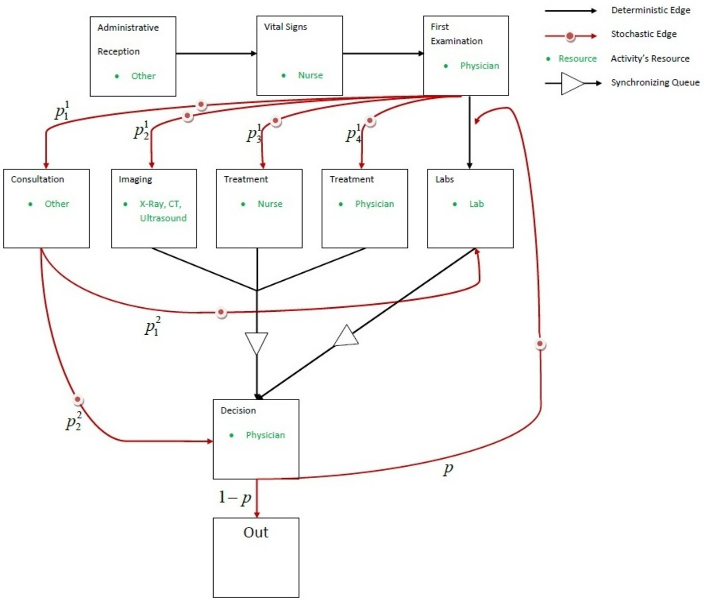
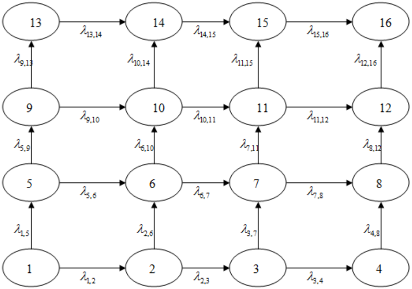


ED JFJ Network



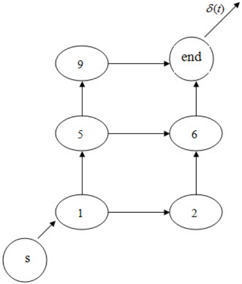
The Black Box Method

Consider a fork join network with V its set of nodes and A its set of arcs. To calculate the Offered Load of station i, one can define a new fork join network i^{-}, as follows:



The Black Box Method

- Let V^- be all the nodes of V which have a directed path to station i.



Example: the network 10^-

The Black Box Method

- Let V^- be all the nodes of V which have a directed path to station i.
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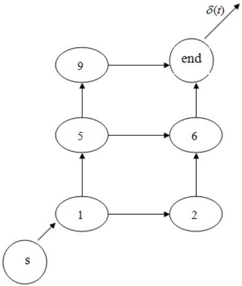

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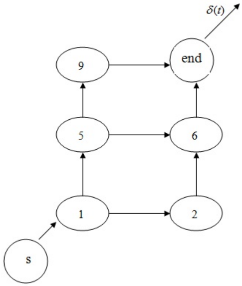
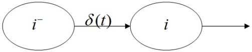
- Let i^- be the fork-join network with V^- its set of nodes and A^- its set of arcs.



Example: the network 10^-

We now have a Tandem of 2 service stations, where queue i is the second queue.



The Offered-Load at station i

The offered load at station i, is given by

$$R_i(t) = E(\lambda(t - T_i - S_i^e)) \cdot E(S_i), \quad t \geq 0,$$

where T_i is the total sojourn time of network i^- and S_i^e is the residual service time at station i.

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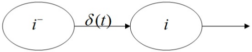

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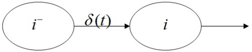

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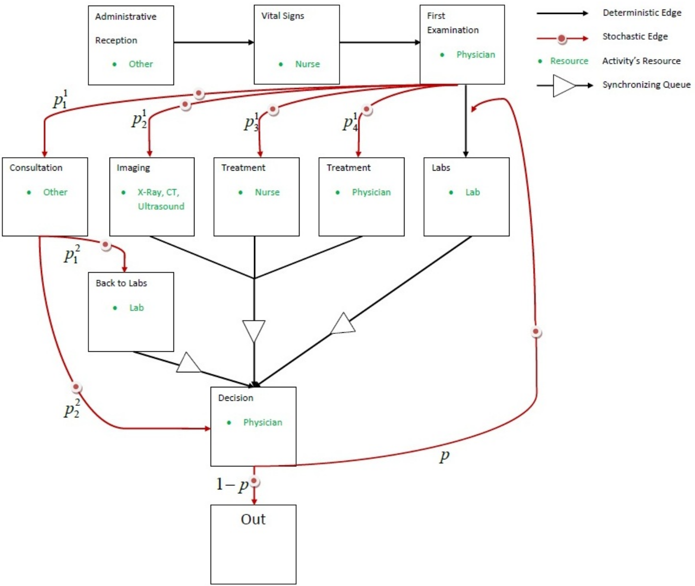
- Thus, finding the distribution of T_i is required for calculating the Offered-Load of station i.
- S_i^e's density function is $f_{S_i^e}(x) = \frac{1-F_{S_i(x)}}{E(S_i)}$.

ED's JFJ Disaggregation



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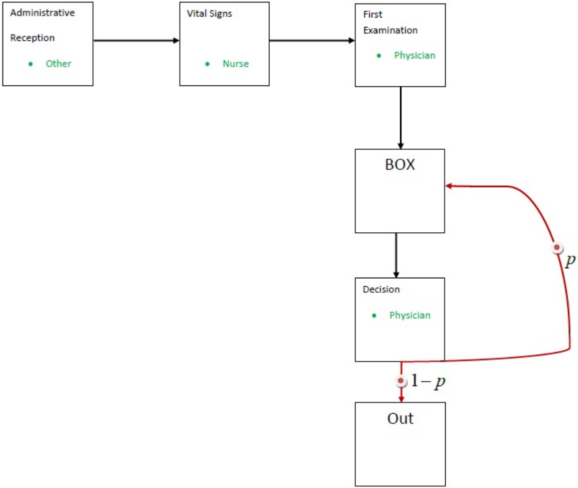

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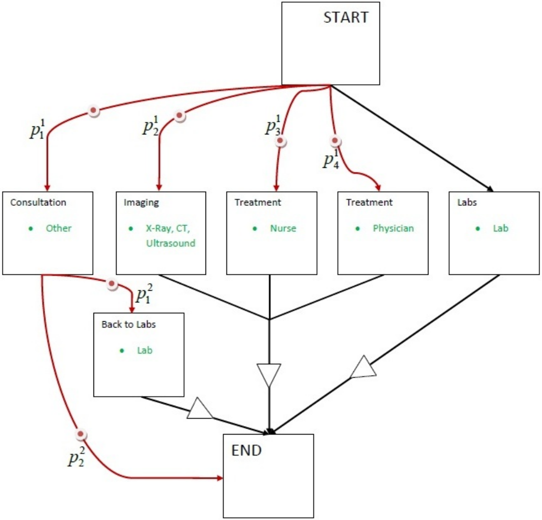
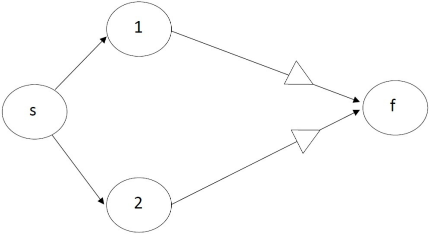
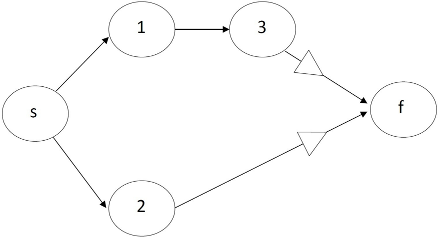


M1 Network



$\psi_{M1}(\theta) = \psi_{S_{1,2}}(\theta) \cdot (\frac{\mu_1}{\mu_1+\mu_2} \cdot \psi_{S_2}(\theta) + \frac{\mu_2}{\mu_1+\mu_2} \cdot \psi_{S_1}(\theta))$,
where $S_{1,2} \sim exp(\mu_1 + \mu_2)$.

M2 Network



$\psi_{M2}(\theta) = \psi_{S_{1,2}}(\theta) \cdot \left(\frac{\mu_1}{\mu_1+\mu_2} \cdot \psi_{S_{2,3}}(\theta) \cdot \{\frac{\mu_2}{\mu_2+\mu_3} \cdot \psi_{S_3}(\theta) + \frac{\mu_3}{\mu_2+\mu_3} \cdot \psi_{S_2}(\theta)\} + \frac{\mu_2}{\mu_1+\mu_2} \cdot \psi_{S_1}(\theta) \cdot \psi_{S_3}(\theta)\right),$
where $S_{1,2} \sim exp(\mu_1 + \mu_2)$,$S_{2,3} \sim exp(\mu_2 + \mu_3)$

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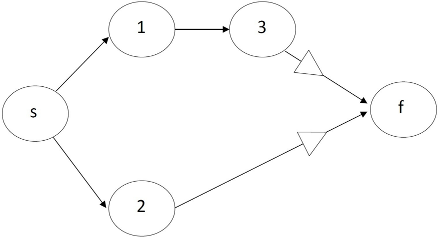

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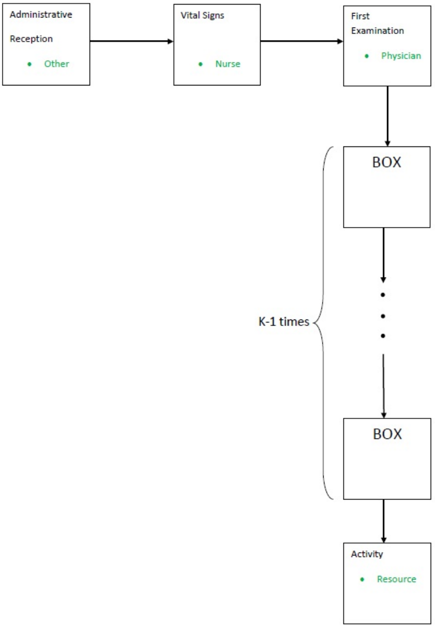
where $S_{1,2} \sim exp(\mu_1 + \mu_2)$, $S_{2,3} \sim exp(\mu_2 + \mu_3)$

- $\psi_{S_{BOX}}(\theta)$ is composed of all the possible routes that a patient can take in the ED

Example: The Characteristic Function of an Activity from "BOX"

The computation of $\psi_a(\theta)$ (a could be treatment, imaging etc.):

- compute $\psi_{a,k}(\theta)$, the characteristic function of patients that visit "BOX" exactly k times


Administrative
Reception
Other
Vital Signs
Nurse
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Examination
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BOX
K-1 times
BOX
Activity
Resource

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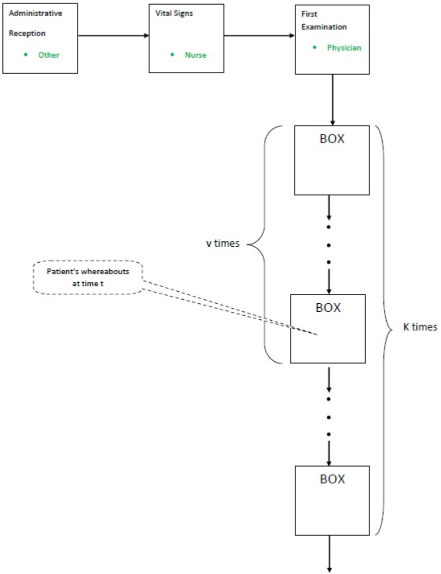

Administrative
Reception
Other
Vital Signs
Nurse
First
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Physician
BOX
v times
Patient's whereabouts
at time t
BOX
K times
BOX

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  $\psi_{S_{AR}} \cdot \psi_{S_{VS}} \cdot \psi_{S_{FE}} \cdot \psi_{S_{a,e}}(\theta)\psi_{S_I}(\theta) p_a \frac{1-\psi^k_{S_{BOX}}(\theta)}{1-\psi_{S_{BOX}}(\theta)}$

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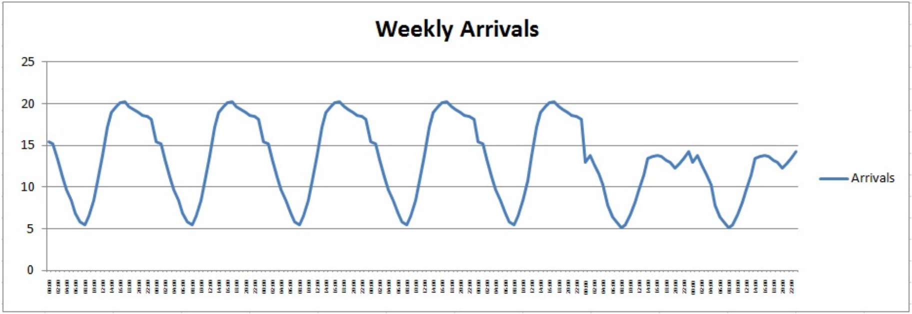
- $\psi_a(\theta) = \sum_{k=1}^{\infty} \psi_{a,k}(\theta) = p_a \cdot \psi_{S_{AR}} \cdot \psi_{S_{VS}} \cdot \psi_{S_{FE}} \cdot \psi_{S_a}(\theta) \cdot \frac{1}{1-p\psi_{S_{BOX}}(\theta)}$

Periodic Arrival Rate: Actual Arrival Rate



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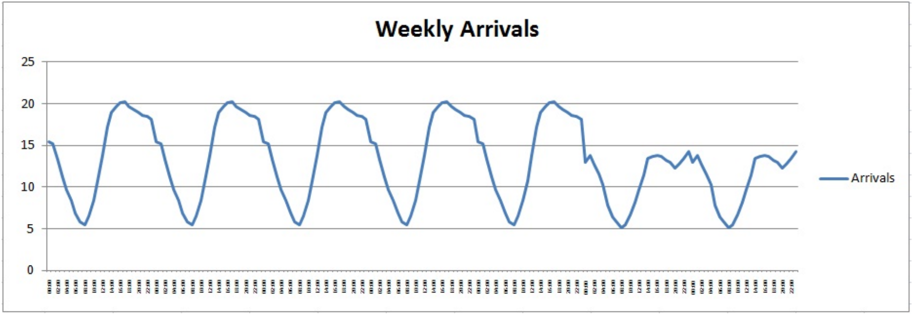

- The periodic nature of arrivals allows one to calculate the Offered-Load in an easier way: **Discrete Fourier transform.**

Periodic Arrival Rate: Actual Arrival Rate

Example: The Offered-Load of Internal Drs:

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1. We look at the activities that are executed by Internal Dr's: "First Examination", "Treatment" and "Decision".
2. We then calculate the Offered-Load of each one of the activities.
3. Summing up the Offered-Load of the activities will give us the Offered-Load of Internal Dr's.

Periodic Arrival Rate: Actual Arrival Rate

- **Example: The Offered-Load of Internal Drs:**

$$R(t) = \sum_{j=1}^{3} R_{a_j}(t) = \sum_{j=1}^{3} \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} \psi_S(\frac{2\pi i}{N}) \cdot E(S_{a_j}),$$

where a_1="First Examination", a_2="Treatment" and a_3="Decision".

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- $x_0, ..., x_{N-1}$ are the arrival rates at times $t_0 = 0, ...t_{N-1} = N - 1$.

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- $x_0, ..., x_{N-1}$ are the arrival rates at times $t_0 = 0, ...t_{N-1} = N - 1$.
- X_k is the discrete Fourier transform of $x_0, ..., x_{N-1}$
  $(X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn} \quad k = 0, ..., N-1)$,
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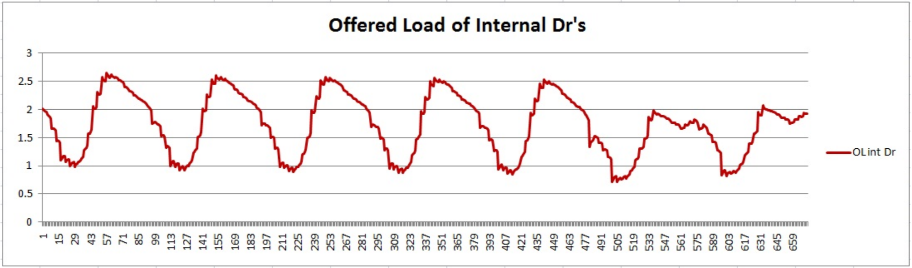
- S_a is the service time of activity a.
- S is the completion time of the network that starts at the ED entry and ends at activity a.

Periodic Arrival Rate: Actual Arrival Rate



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$$N(t) = round(R(t) + \beta \cdot \sqrt{R(t)}) \ ,$$

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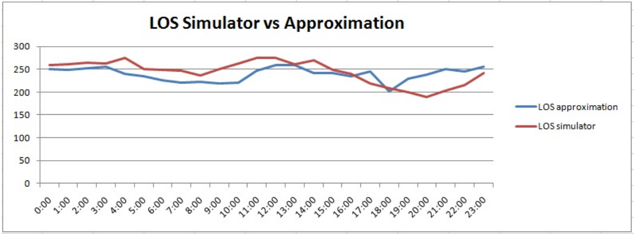
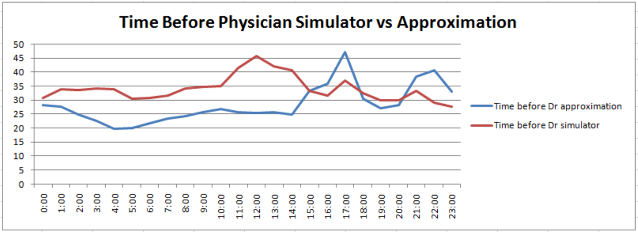
- $round(x) = \{ \begin{smallmatrix} \lceil x \rceil & if & \lceil x \rceil - x \geq 0.5 \\ \lfloor x \rfloor & if & \lceil x \rceil - x < 0.5 \end{smallmatrix}$.

Comparing results: Simulator vs Approximation

For example, results of internal Drs (staffing with $\beta = 0.7$):





Project Completion Time by Continues Time Markov Chain

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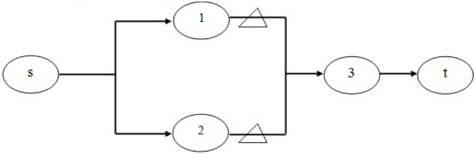
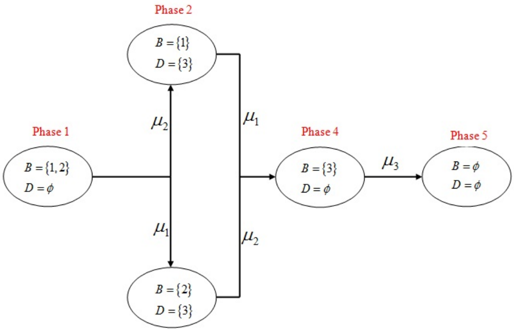
- The continues time Markov chain state space consists of all the pairs (B, D) : B denote all the <u>active</u> nodes and D all the <u>dormant</u> nodes.

Project Completion Time: Example

The fork-join Network: (here, μ_i is the service rate of station i)



The CTMC rate diagram:



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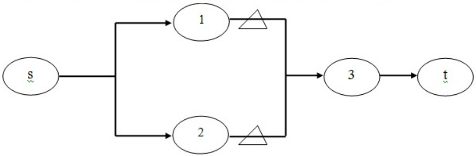

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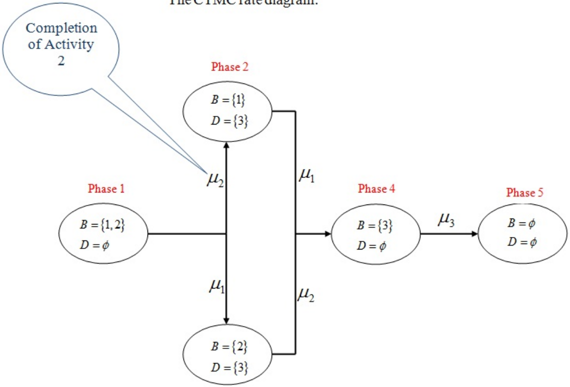

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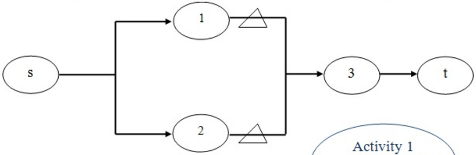

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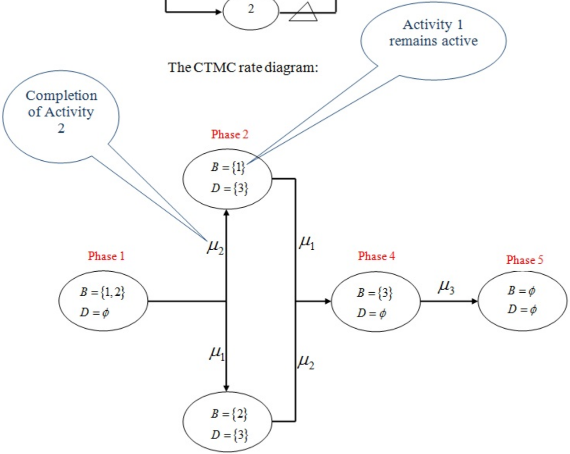

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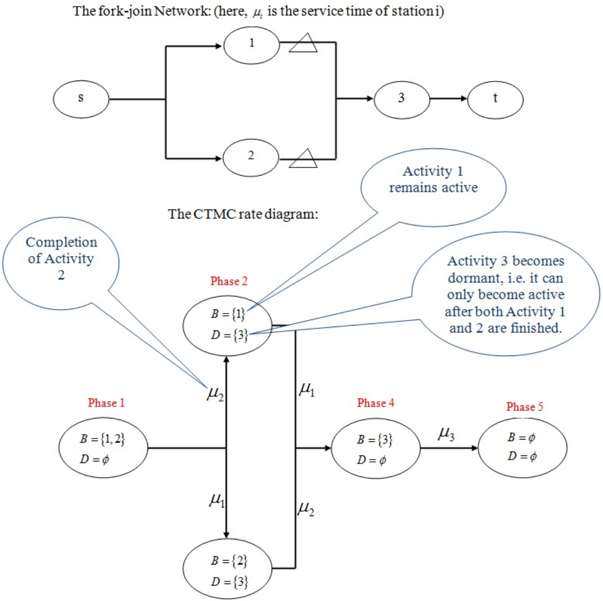

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- We start the algorithm with $p_N(t) \equiv 1$, for $t \geq 0$ and compute $p_{N-1}(t), ..., p_1(t), p_0(t)$ recursively backward.

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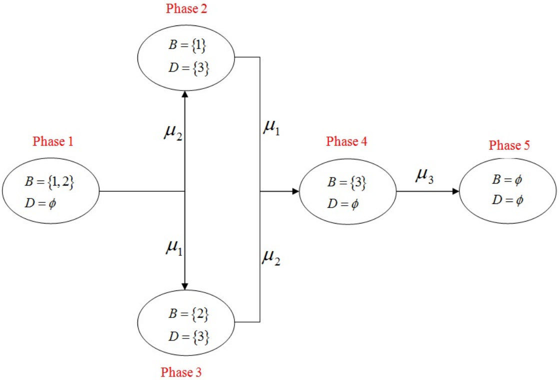

- $P_5(t) = 1$,
- $P_3'(t) = \mu_2 \cdot P_4(t)$,
- $P_1'(t) = \mu_1 \cdot P_3(t) + \mu_2 \cdot P_2(t)$.
- $P_4'(t) = \mu_3 \cdot P_5(t)$,
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- Staffing according to the Offered Load stabilizes measures of performance of the emergency department over time.
- Tractable analysis of a complex network (fork-join) captures the full complexity of an emergency department.


Questions
Thank You