QED Q's:

Quality- and Efficiency-Driven Queues

with a focus on Call/Contact Centers

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Technion, Haifa, Israel

http://ie.technion.ac.il/serveng

Columbia University, September 2008

Research Partners

- Empirical/Statistical Analysis:
  Brown, Gans, Sakov, Shen, Zeltyn, Zhao
- Students:
  Garnett, Gurvich, Aldor, Khudiakov, Feldman, Rosenshmidt, Yom-Tov, Marmor, Tseytlin, . . .
- Colleagues:
  Armony, Jelenkovic, Massey, Momcilovic, Whitt, . . .
- Technion SEE Lab:
  Feigin, Trofimov, Nadjharov, Gavako, Liberman; RA's, . . .

Contents

- On Service Science / Engineering / **QED Q's**
- Example: Anatomy of "Waiting for Service"
- The Basic (Operational) Call-Center Model: **Palm/Erlang-A** (M/M/N+M)
- Validating Erlang-A? All **Assumptions Violated**
- But **Erlang-A Works! Why?** Framework - Asymptotic Regimes: **QED, ED, ED+QED**
- Explain **Practice**: "Right Answers for the Wrong Reasons"
- Technion's **SEE (Service Enterprise Engineering) Lab**: **DataMOCCA, SEEStat**

Main Messages

1. **Simple Models** at the Service of **Complex Realities**.

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5. Service Science / Management / Engineering are emerging **Academic Disciplines**. For example, universities and

USA NSF (SEE), IBM (SSME), Germany IAO (ServEng), ...

Background Material (Downloadable)

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- Trofimov, Feigin, M., Ishay, Nadjharov: "DataMOCCA: Models for Call/Contact Center Analysis." Technion Report, 2004-2006.
- M. "Call Centers: Research Bibliography with Abstracts." Version 7, December 2006.

Queueing Science: Data-Based QED's Q's

Traditional Queueing Theory predicts that **Service-Quality** and **Servers' Efficiency** **must** be traded off against each other.

For example, **M/M/1 in heavy-traffic**: **91%** server's utilization goes with

$$\textbf{Congestion Index} = \frac{E[Wait]}{E[Service]} = \textbf{10},$$

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Yet, heavily-loaded queueing systems with **Congestion Index = 0.1** (Waiting one order of magnitude less than Service) are prevalent:

- **Call Centers**: Wait **"seconds"** for **minutes** service;
- **Transportation**: Search **"minutes"** for **hours** parking;
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and, moreover, a significant fraction are not delayed in queue. (For example, in well-run call-centers, **50%** served "immediately", along with over **90%** agents' utilization, is not uncommon) **?**

Prerequisite: Data

Averages Prevalent.

But I need data at the level of the **Individual Transaction**: For each service transaction (during a phone-service in a call center, or a patient's stay in a hospital), its **operational history** = time-stamps of events.

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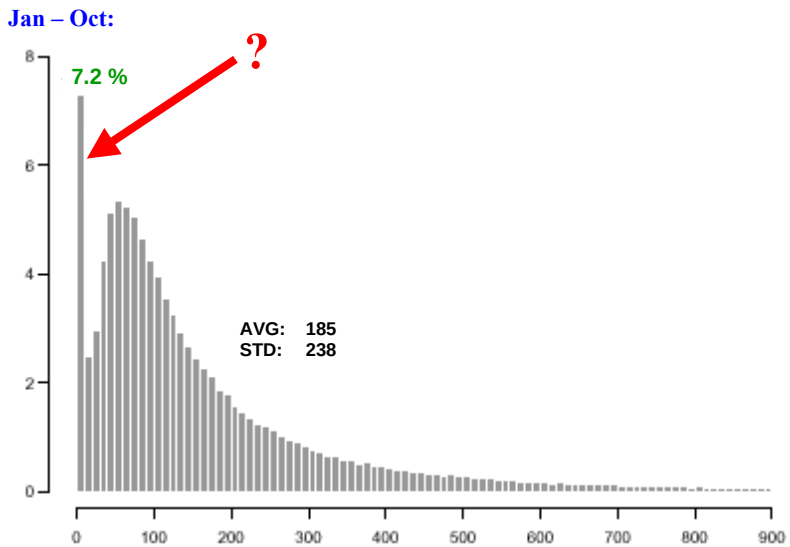
Sources: **"Service-floor"** (vs. Industry-level, Surveys, . . .)

- Administrative (Court, via "paper analysis")
- Face-to-Face (Bank, via bar-code readers)
- Telephone (Call Centers, via ACD / CTI)
- Expanding:
  - Hospitals (via RFID)
  - IVR (VRU), internet, chat (multi-media)
  - Operational + Financial + Marketing / Clinical history

Beyond Averages (+ The Human Factor)

Histogram of Service Times in an Israeli Call Center

January-October



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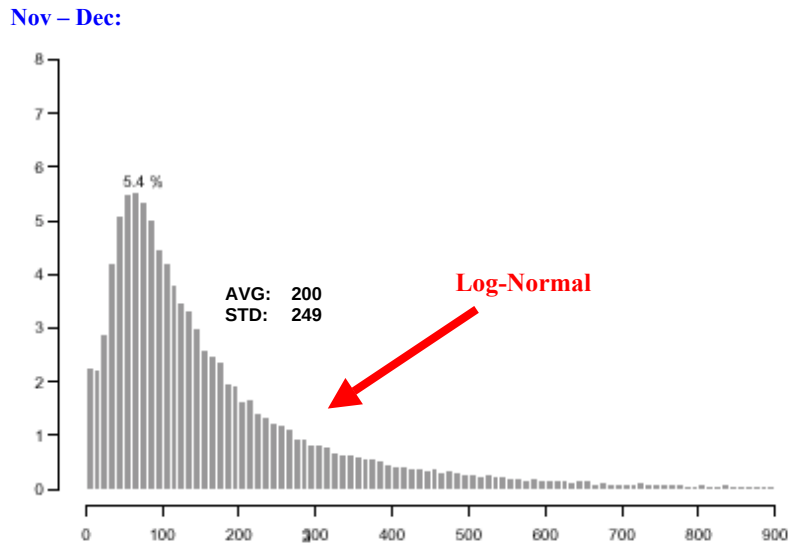

- **7.2% Short-Services:**

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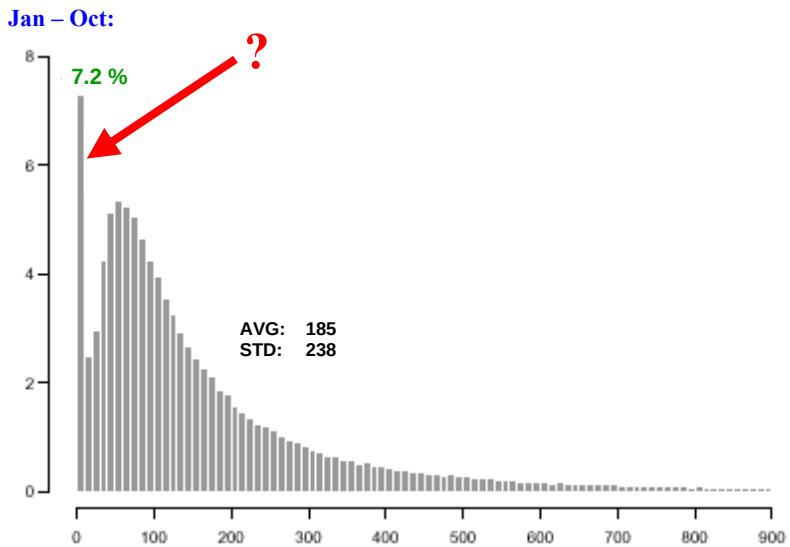

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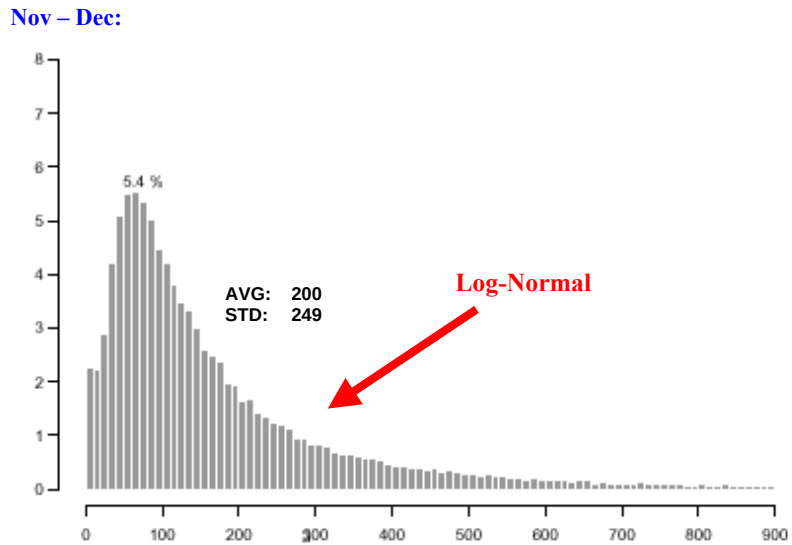


- **7.2% Short-Services:** Agents' "Abandon" (improve bonus, rest)
- **Distributions**, not only Averages, must be measured.
- **Lognormal** service times prevalent in call centers (Why?)

Present Focus: Call Centers

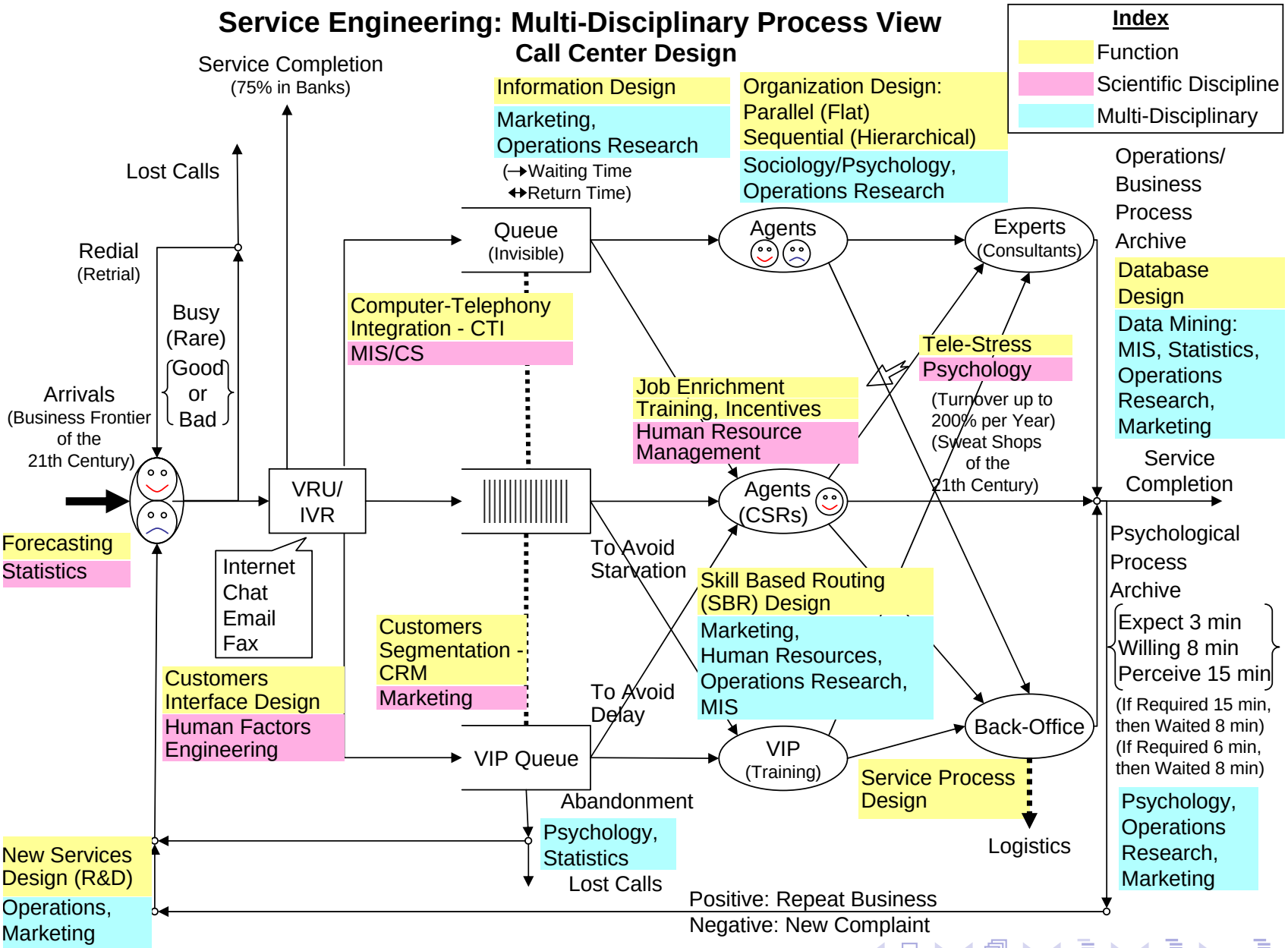
U.S. Statistics (Relevant Elsewhere)

- Over 60% of annual business volume via the telephone
- 100,000 – 200,000 call centers
- 3 – 6 million employees (**2% – 4% workforce**)
- 1000's agents in a "single" call center = 70 % costs.
- 20% annual growth rate
- $200 – $300 billion annual expenditures

Call-Center Environment: Service Network

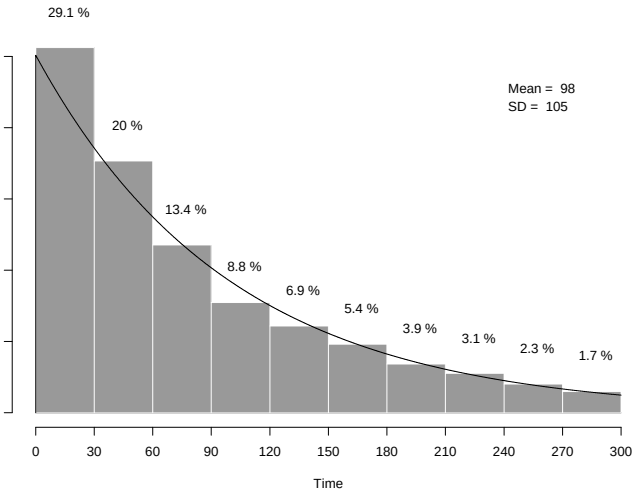
Call-Centers: “Sweat-Shops of the 21st Century"

Call-Center Network: Gallery of Models

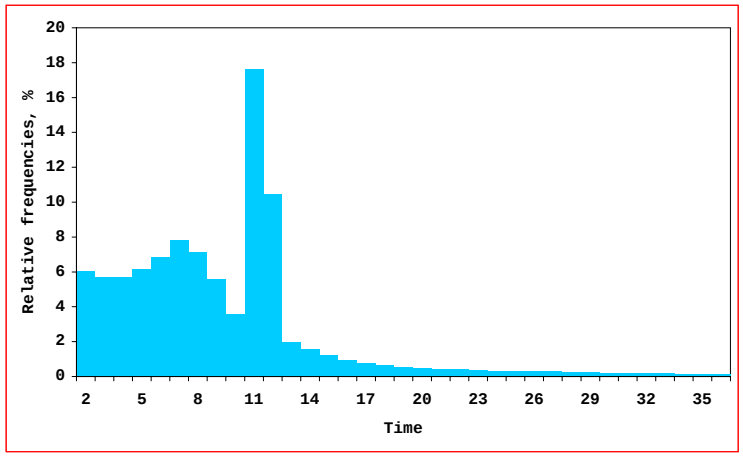


Beyond Averages: Waiting Times in a Call Center

Small Israeli Bank



Large U.S. Bank



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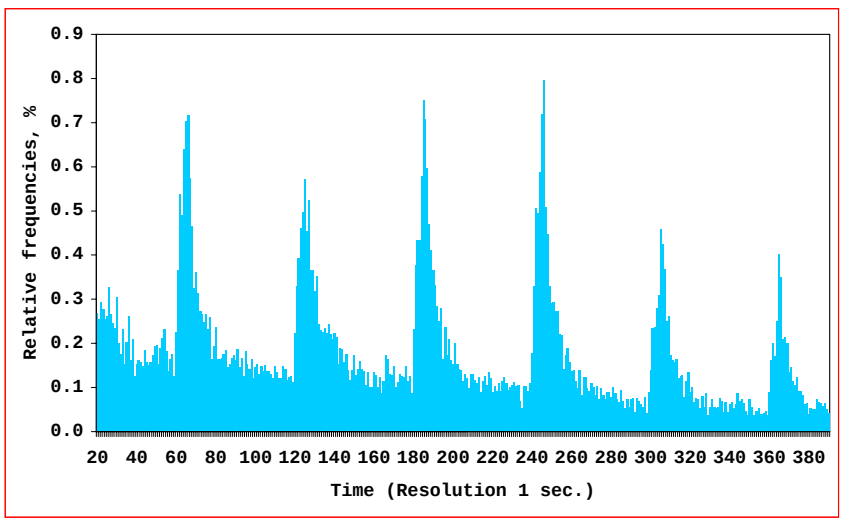

The “Anatomy of Waiting" for Service

Common Experience:

- Expected to wait 5 minutes, Required to 10,
- Felt like 20, Actually waited 10,
- . . . etc.

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An attempt at “Modeling the Experience":

1.	Time that a customer	**expects** to wait	
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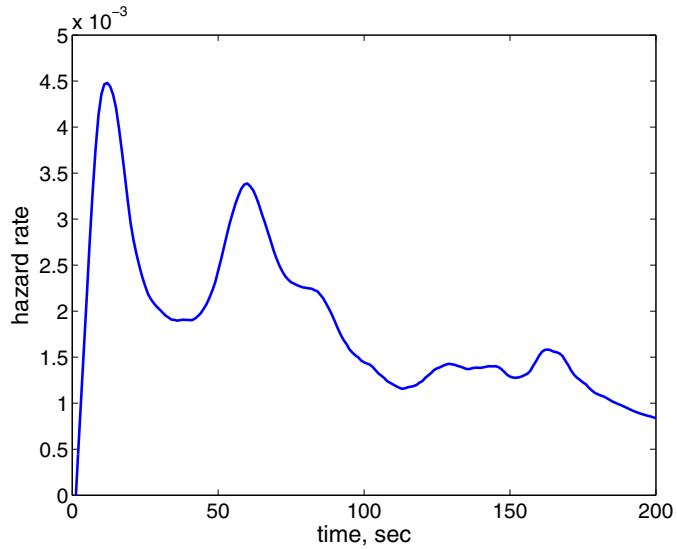
Experienced customers	$\Rightarrow$	Expected = Required
“Rational" customers	$\Rightarrow$	Perceived = Actual.

Then left with (τ, V).

Call Center Data: Hazard Rates (Un-Censored)

(Im)Patience Time τ

Israel



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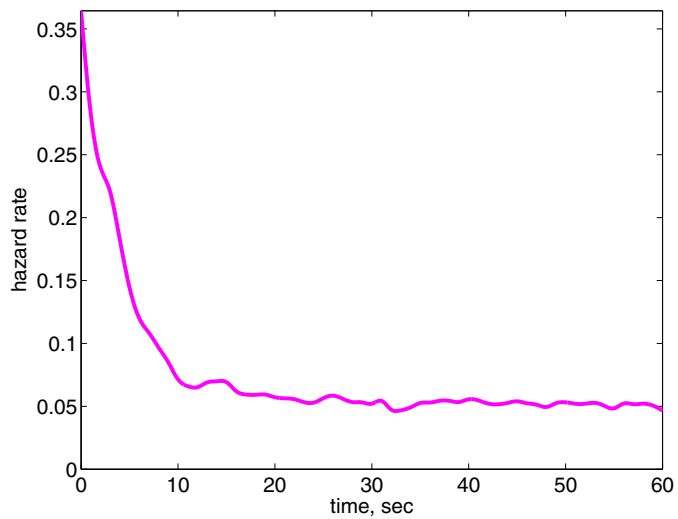

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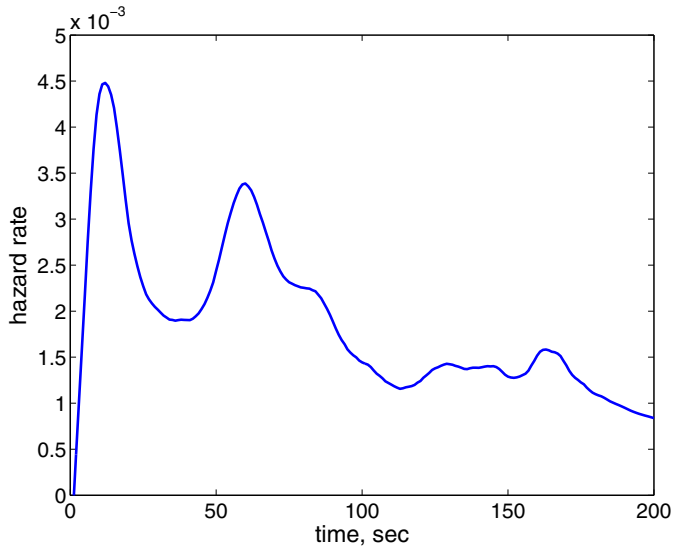

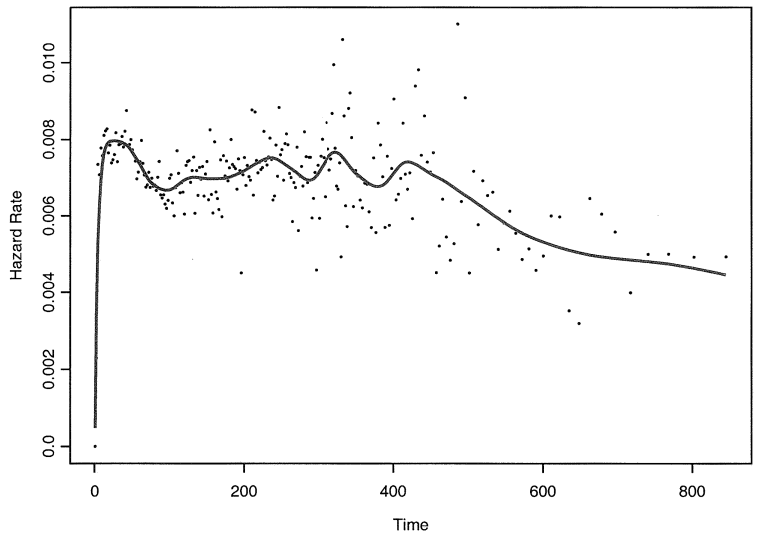

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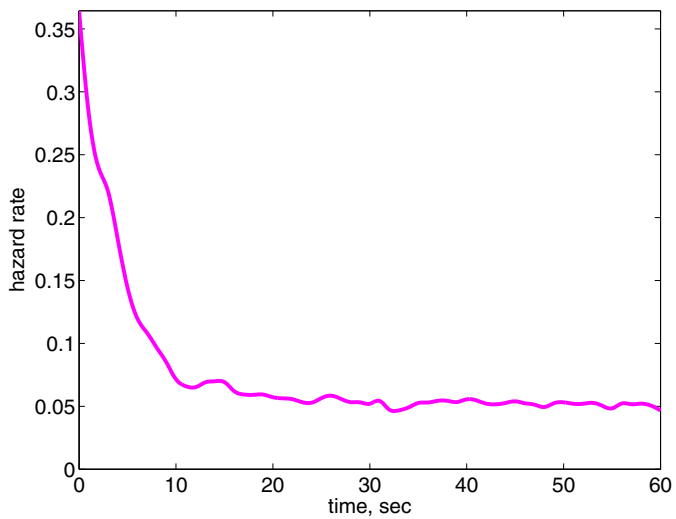

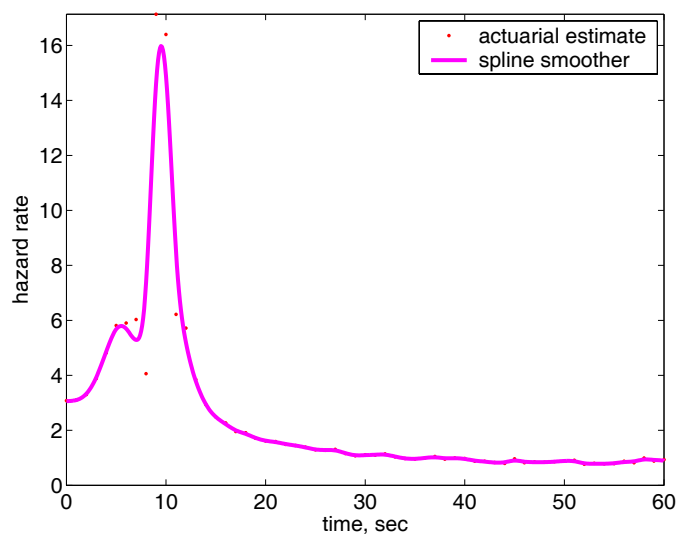

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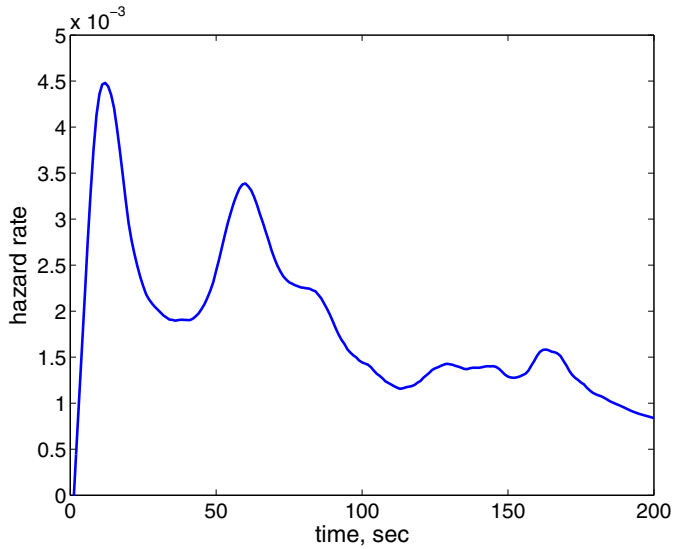

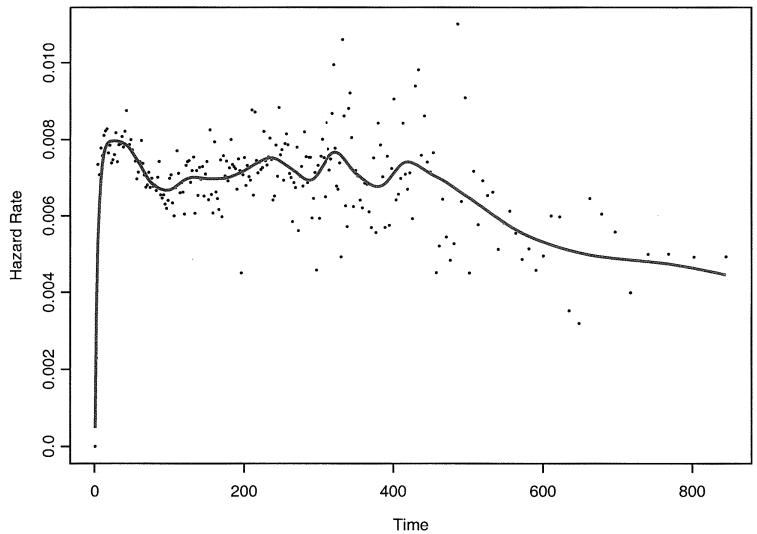

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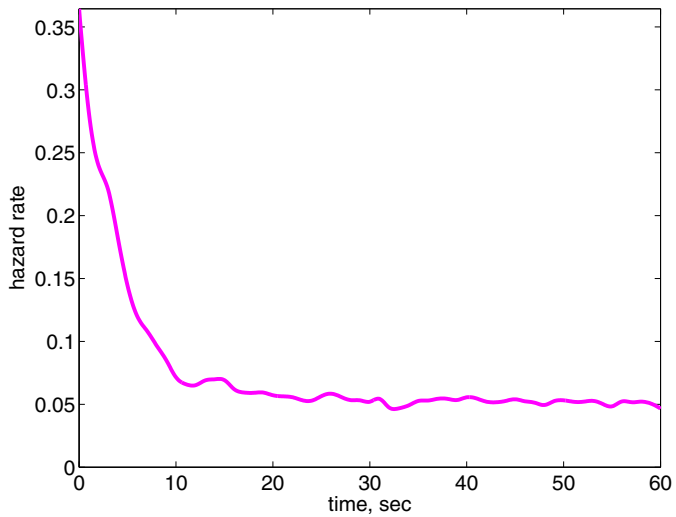
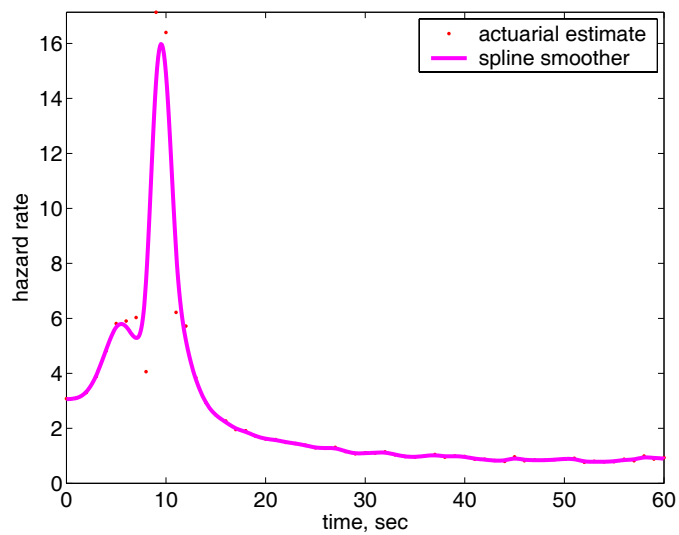
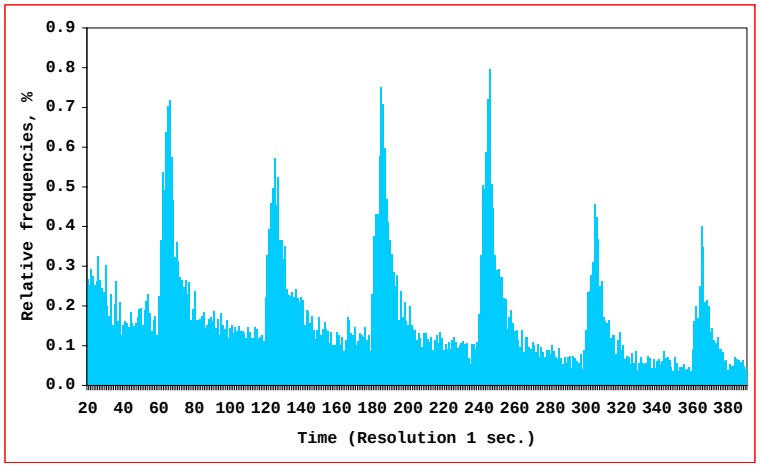




Note: 5% abandoning $\Rightarrow$ 95% (im)patience-observations **censored** !

A “Waiting-Times" Puzzle at a Large Israeli Bank



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- Human: **Voice-announcement** every 60 seconds.

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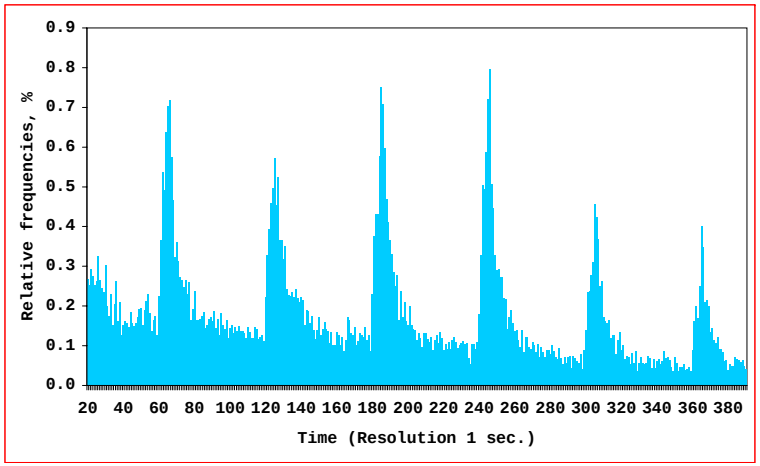

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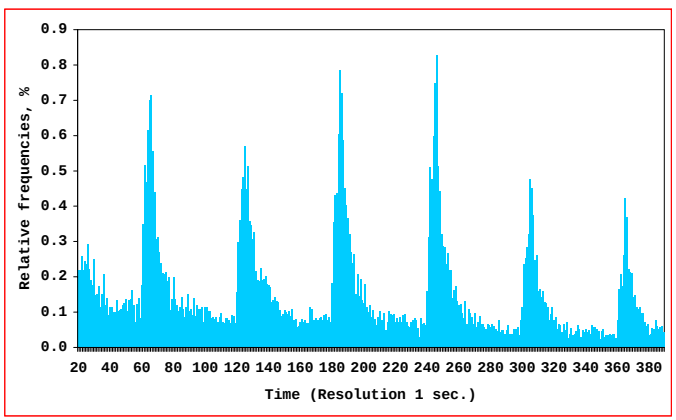
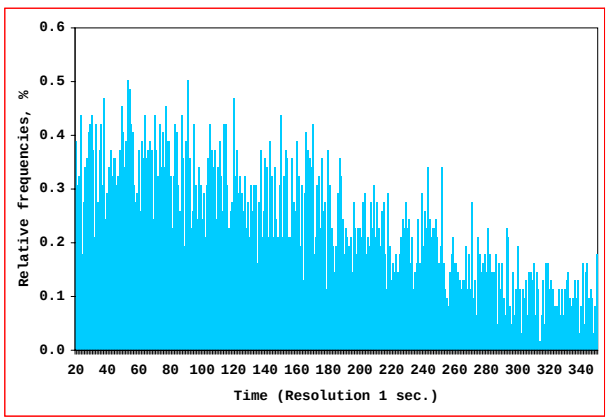
- Human: **Voice-announcement** every 60 seconds.
- System: **Priority-upgrade** (unrevealed) every 60 sec’s (Theory?)

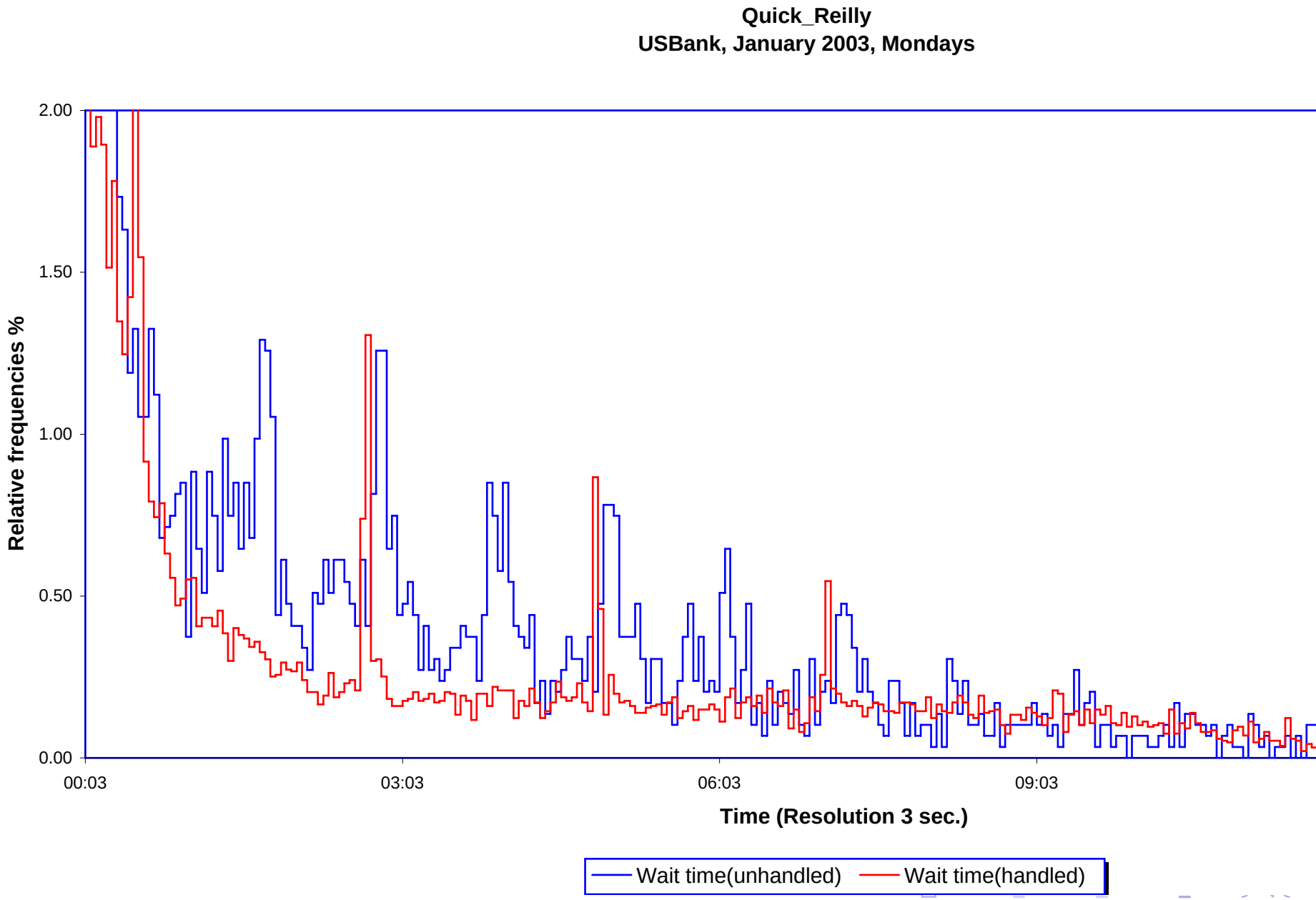
Served Customers



Abandoning Customers



Still a Puzzle at a U.S. Bank



Models for Performance Analysis

- **(Im)Patience**: r.v. τ = Time a customer is **willing to wait**
- **Offered-Wait**: r.v. V = Time a customer is **required to wait**
  (= Waiting time of a customer with infinite patience).
- **Abandonment** $= \{\tau \leq V\}$
- **Service** $= \{\tau > V\}$
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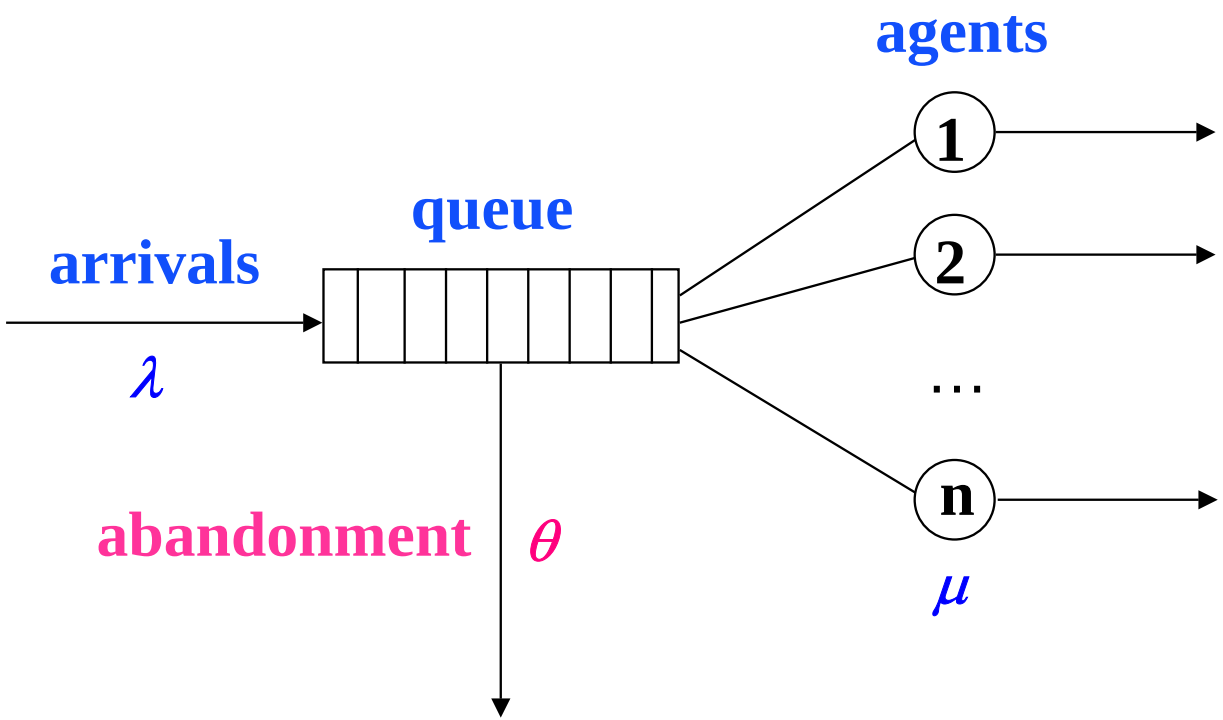
Modeling: τ = **input** to the model, V = **output**.

Operational Performance-Measure calculable in terms of (τ, V):

- eg. **Avg. Wait** = $\mathrm{E}[\min\{\tau, V\}]$ ($\mathrm{E}[W_q|\text{Served}] = \mathrm{E}[V|\tau > V]$)
- eg. **% Abandon** = $\mathrm{P}\{\tau \leq V\}$ ($\mathrm{P}\{5 \text{ sec} < \tau \leq V\}$)

Application: **Staffing – How Many Agents?** (then: When? Who?)

The Basic Staffing Model: Erlang-A (M/M/N + M)



Erlang-A (Palm 1940's) = Birth & Death Q, with parameters:

- λ – **Arrival** rate (Poisson)
- μ – **Service** rate (Exponential)
- θ – **Impatience** rate (Exponential)
- n – Number of **Service-Agents**.

Testing the Erlang-A Primitives

- **Arrivals**: Poisson?
- **Service-durations**: Exponential?
- **(Im)Patience**: Exponential?

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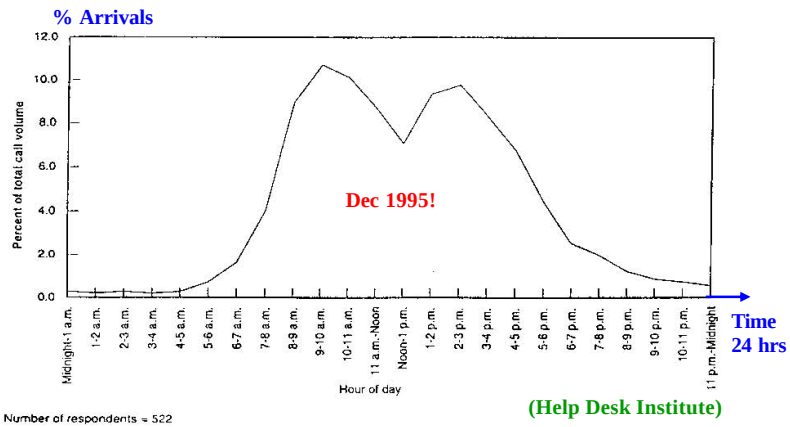
- Primitives independent?
- Customers / Servers Heterogeneous?
- Service discipline FCFS?
- . . . ?

Validation: Support? Refute?

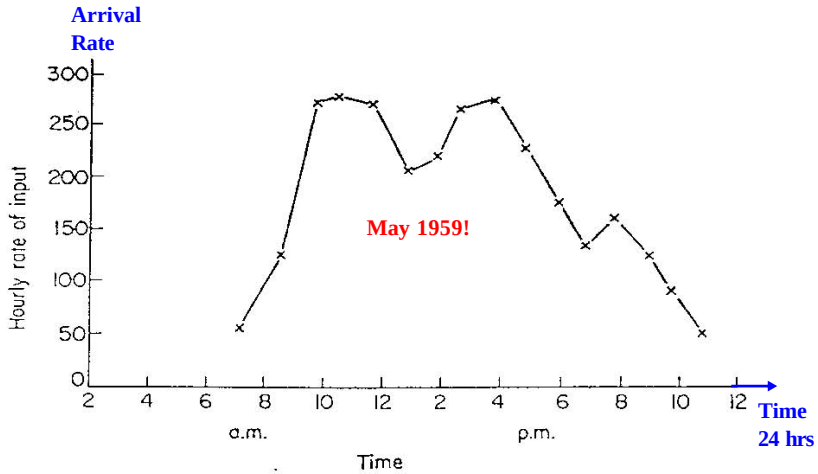
Arrivals to Service: only Poisson-Relatives

Arrival Rate to Three Call Centers

Dec. **1995** (U.S. 700 Helpdesks)



May **1959** (England)



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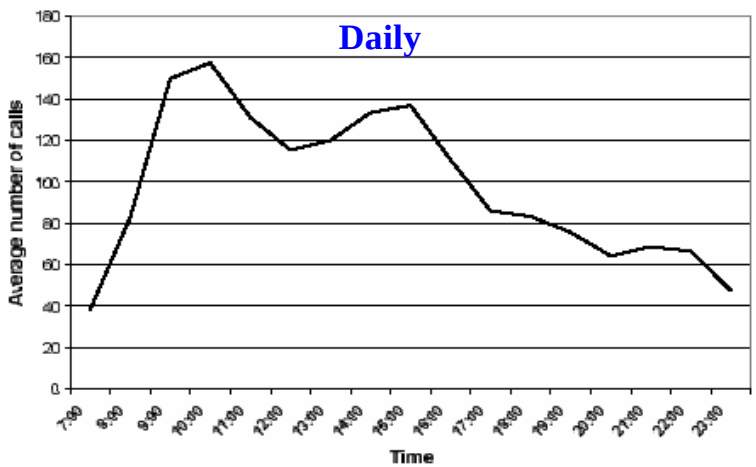

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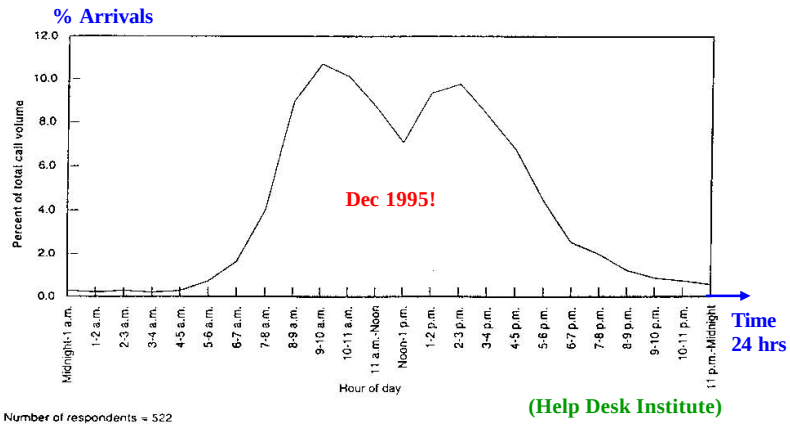

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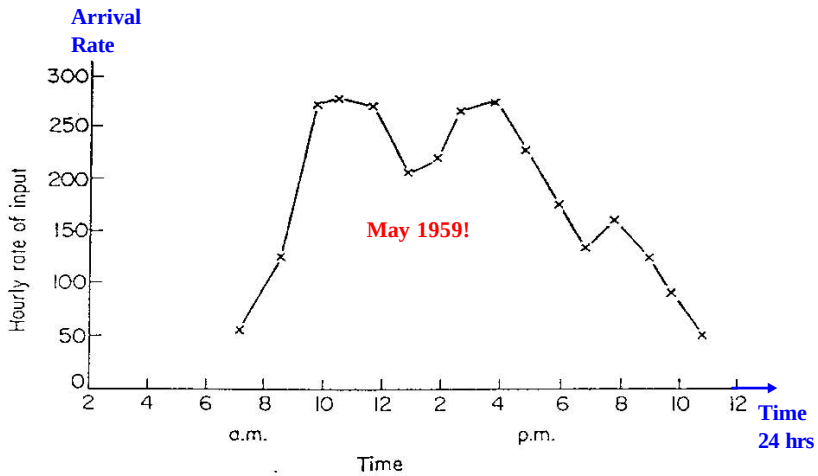

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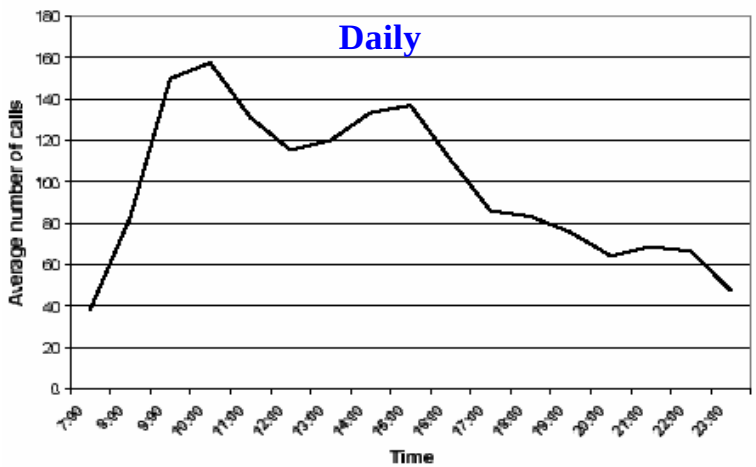
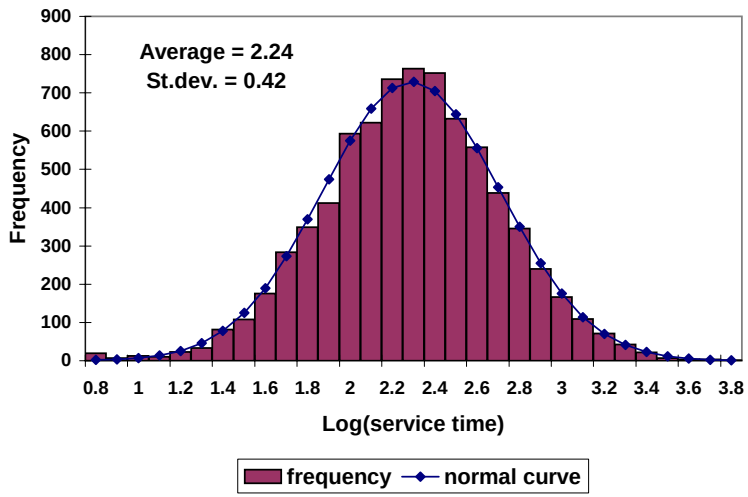


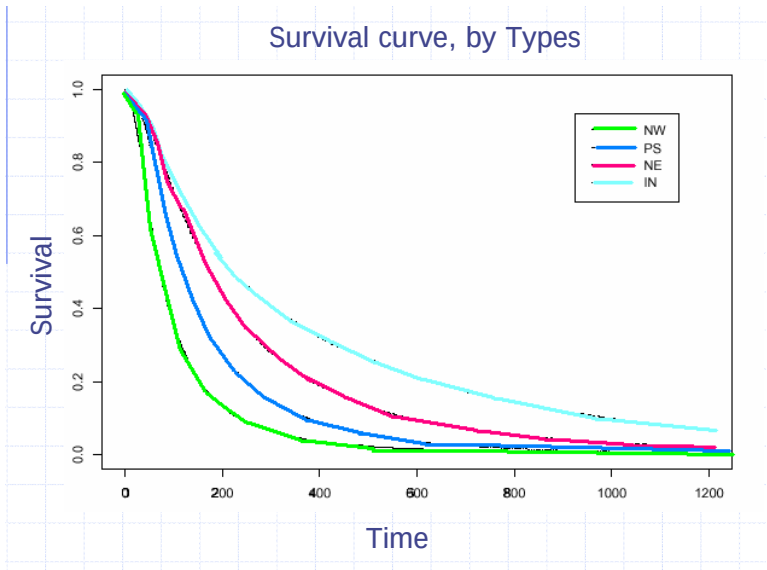
Observation:
Peak Loads at **10:00 & 15:00**

Service Durations: LogNormal Prevalent

Israeli Bank Log-Histogram



Survival-Functions by Service-Class



- **New** Customers: 2 min (NW);
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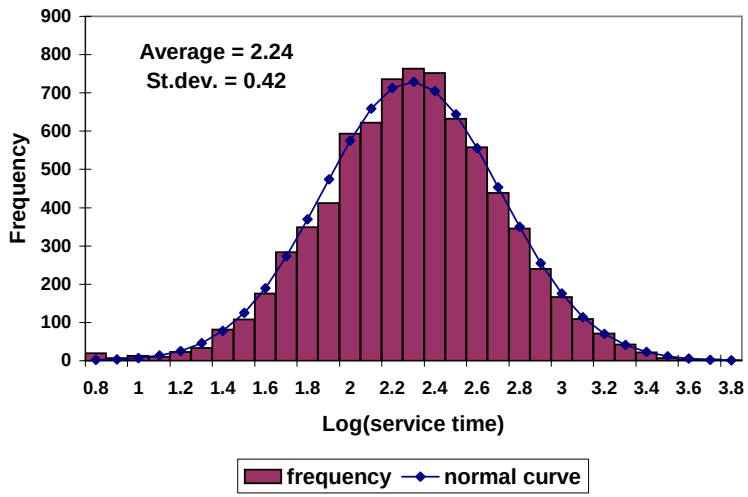

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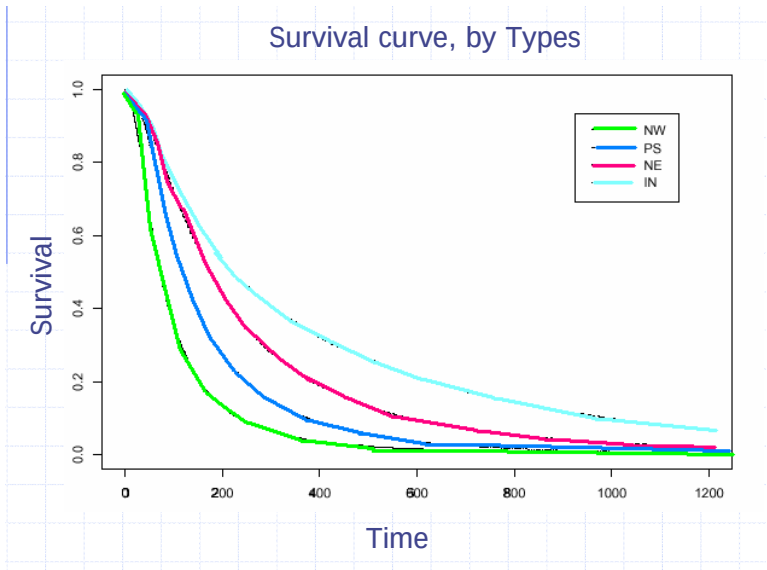

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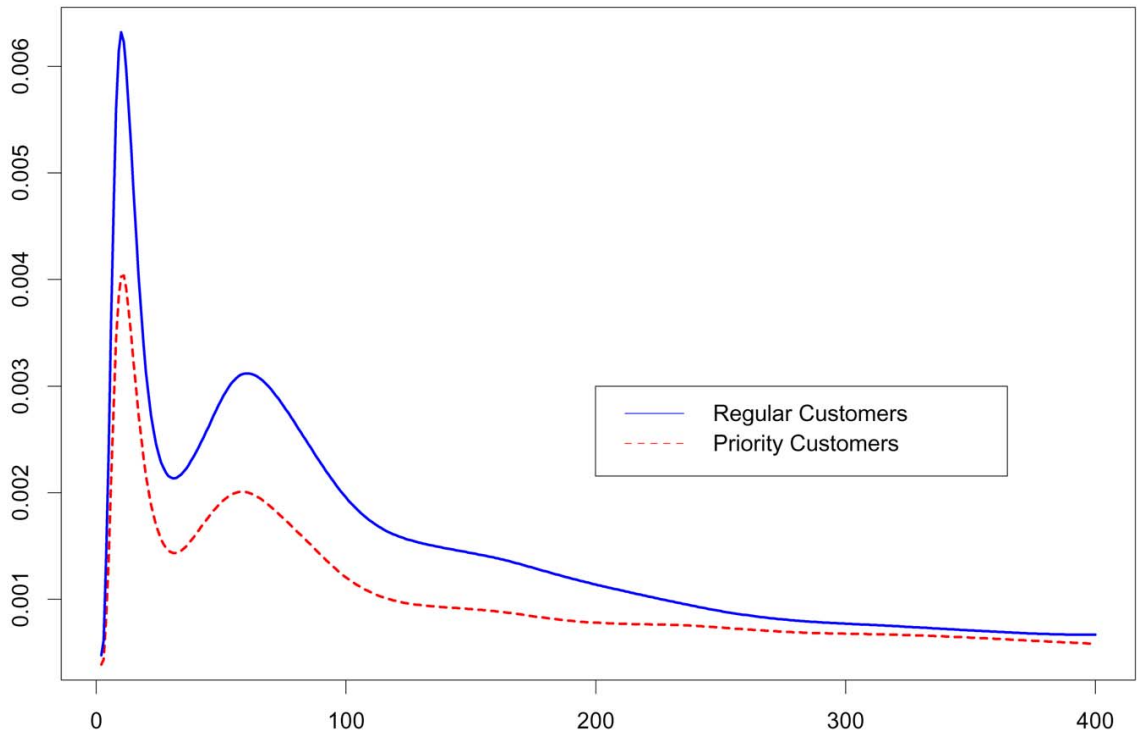
Observation: **VIP** require **longer service** times.

(Im)Patience while Waiting (Palm 1943-53)

Irritation $\propto$ Hazard Rate of (Im)Patience Distribution
Regular over VIP Customers – Israeli Bank



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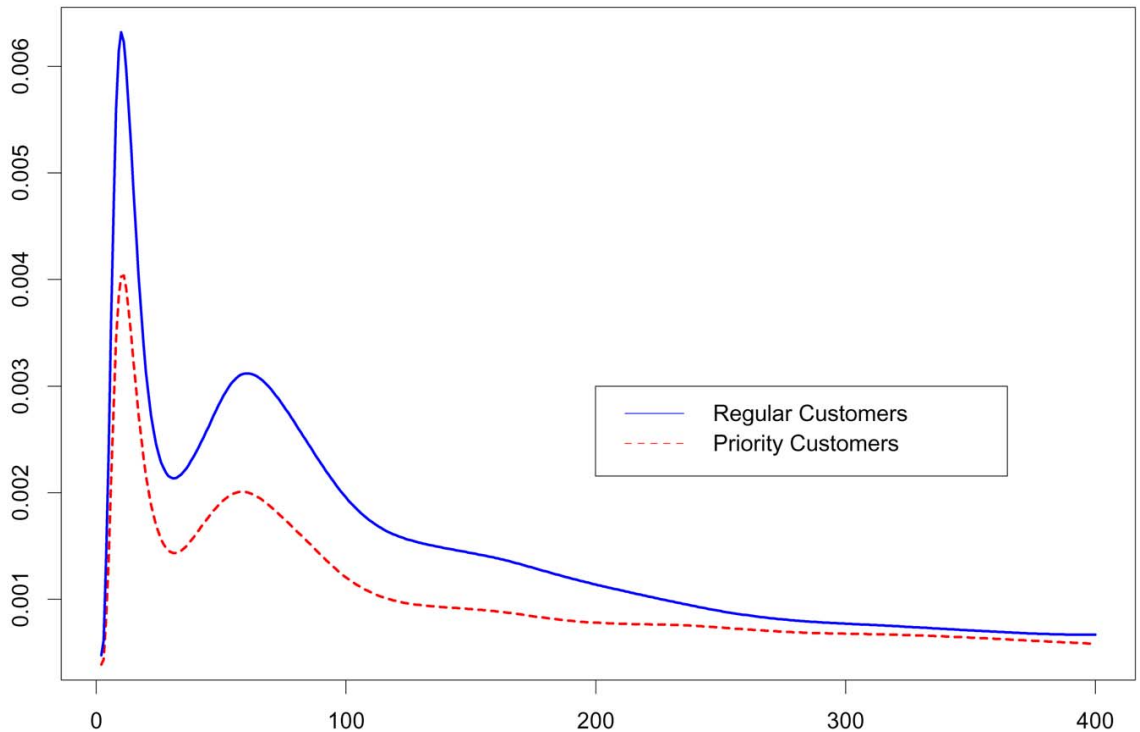

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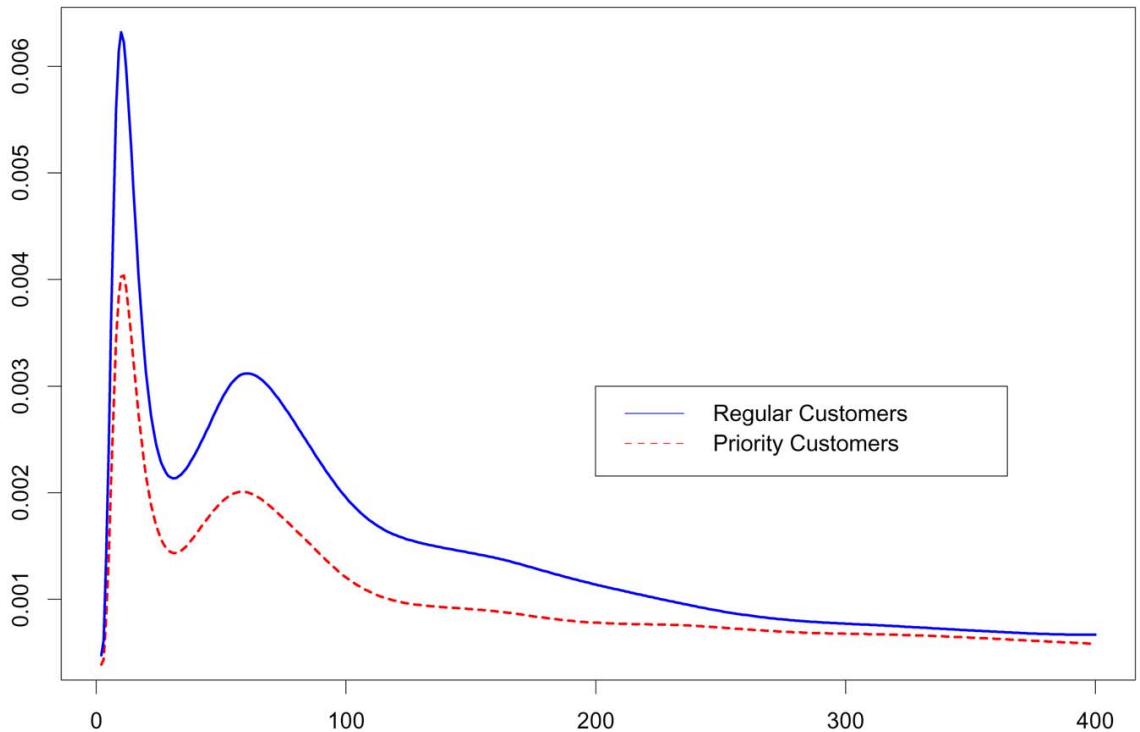

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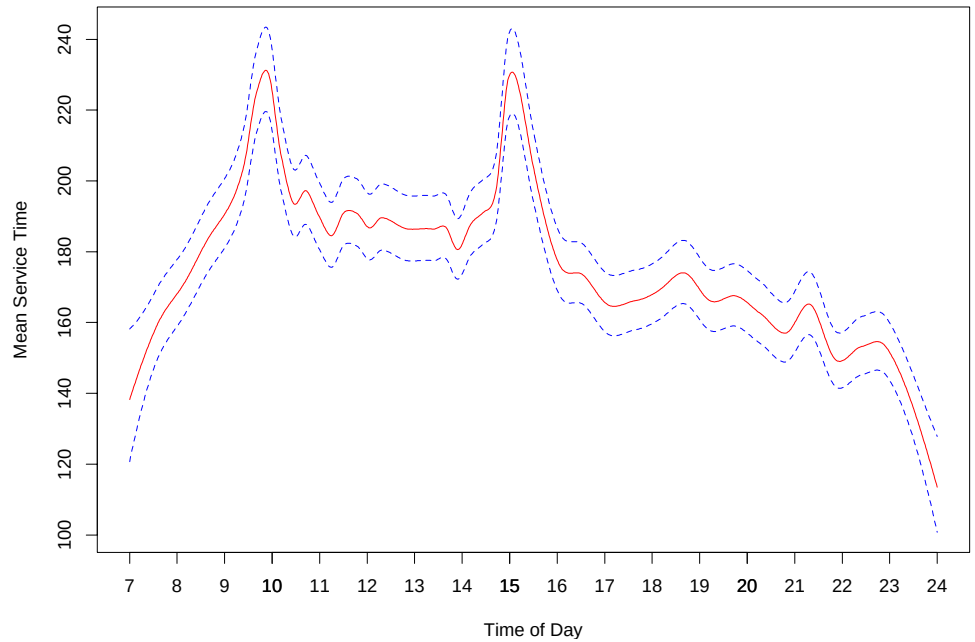
Observation: **VIP** are **more patient** (Needy)

A “Service-Time" Puzzle at an Israeli Bank

Inter-related Primitives

Average Service Time over the Day – Israeli Bank



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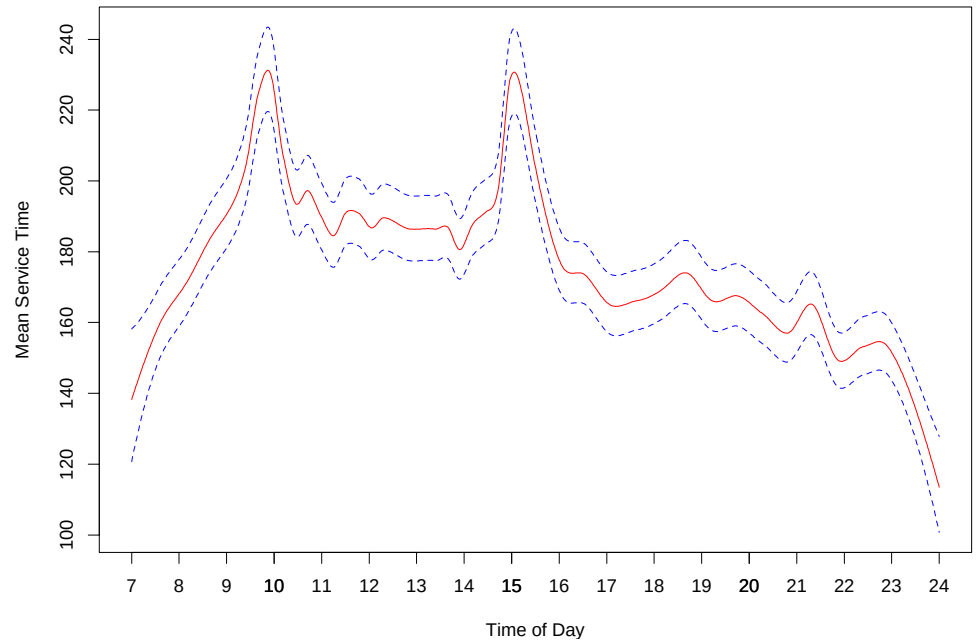

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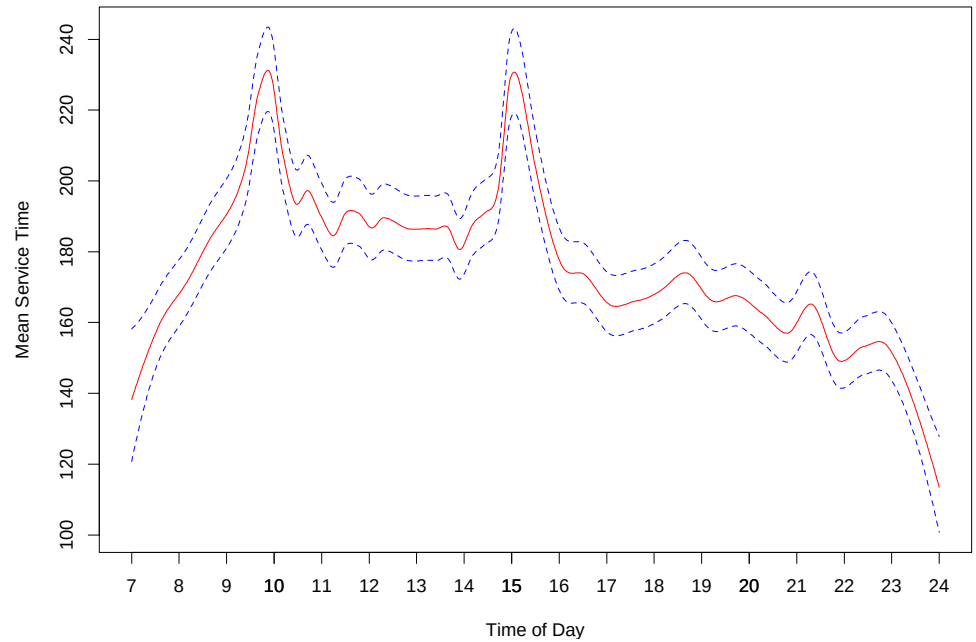

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- Common: Service protocol different (longer) during peak times.
- Operational: The needy abandon less during peak times; hence the **VIP remain** on line, with their **long service** times.

Erlang-A: Practical Relevance?

Experience:

- Arrival process **not pure Poisson** (time-varying, σ^2 too large)
- Service times **not Exponential** (typically close to LogNormal)
- Patience times **not Exponential** (various patterns observed).

- Building Blocks need **not be independent** (eg. long wait possibly implies long service)
- Customers and Servers **not homogeneous** (classes, skills)
- Customers return for service (after busy, abandonment)
- $\cdots$, and more.

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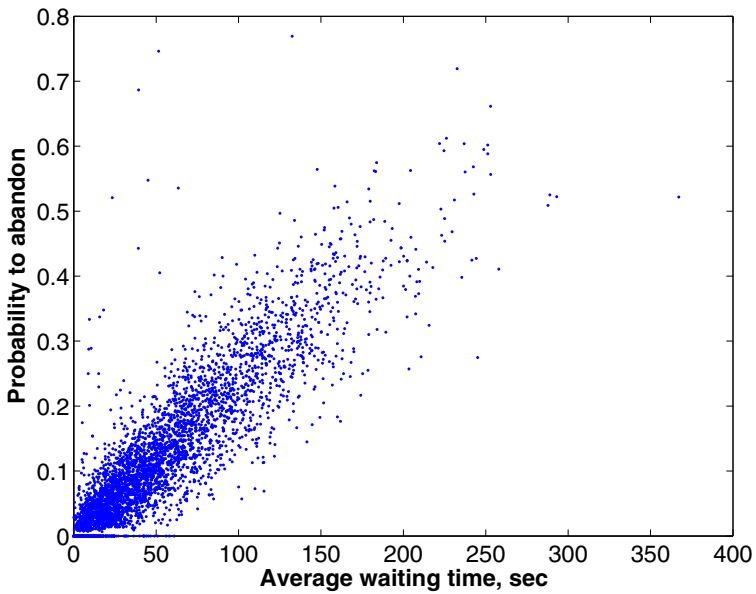
Question: **Is Erlang-A Practically Relevant?**

Estimating (Im)Patience: via P{Ab} $\propto$ E[W_q]

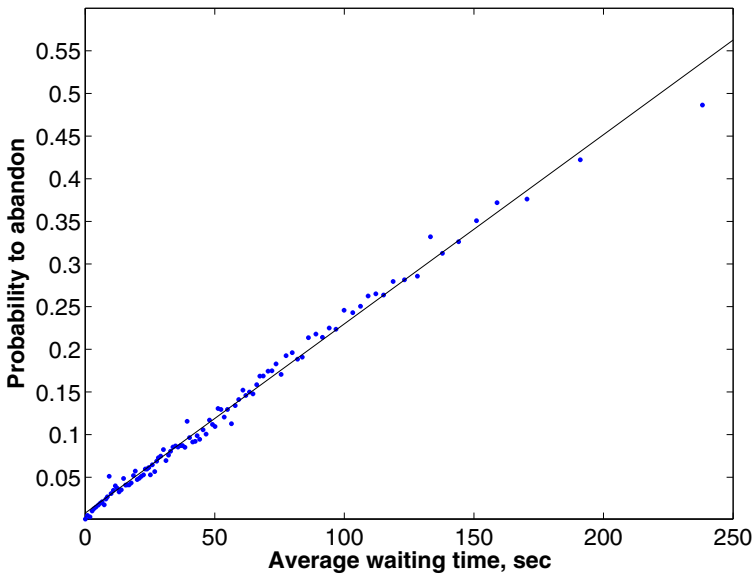
Assume **Exp(θ)** (im)patience. Then, $\mathrm{P\{Ab\}} = \theta \cdot \mathrm{E}[W_q]$.

Israeli Bank: Yearly Data

Hourly Data



Aggregated



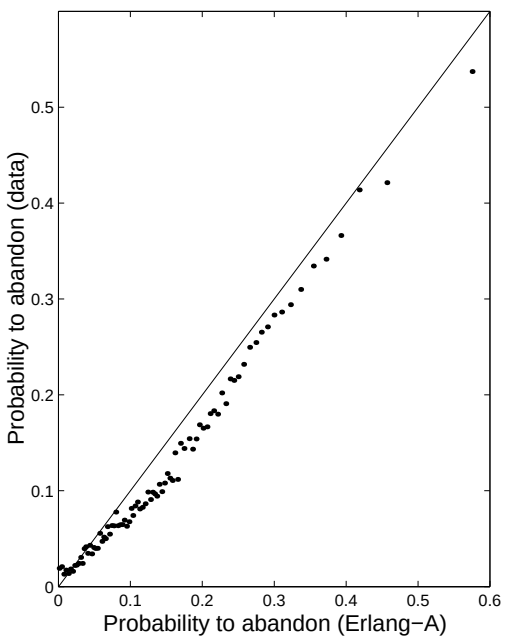
Graphs based on 4158 hour intervals.

Estimate of mean (im)patience: 250/0.55 $\approx$ **450 seconds**.

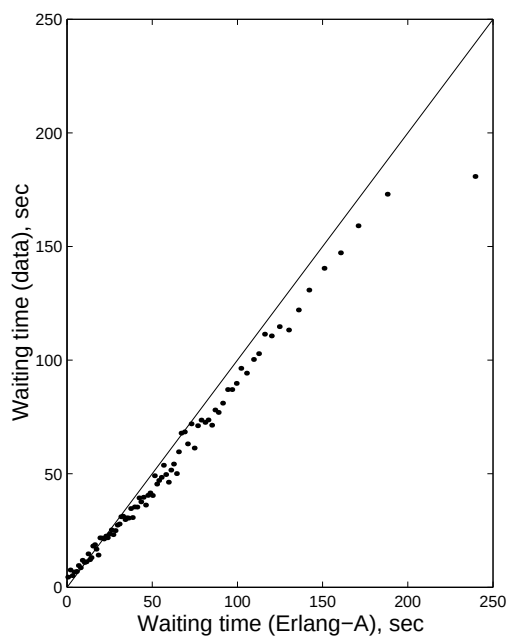
Erlang-A: Fitting a Simple Model to a Complex Reality

- **Small Israeli Banking Call-Center** (10 agents)
- (Im)Patience (θ) estimated via P{Ab} / E[W_q]
- Graphs: **Hourly Performance vs. Erlang-A Predictions**, during 1 year (aggregating groups with 40 similar hours).

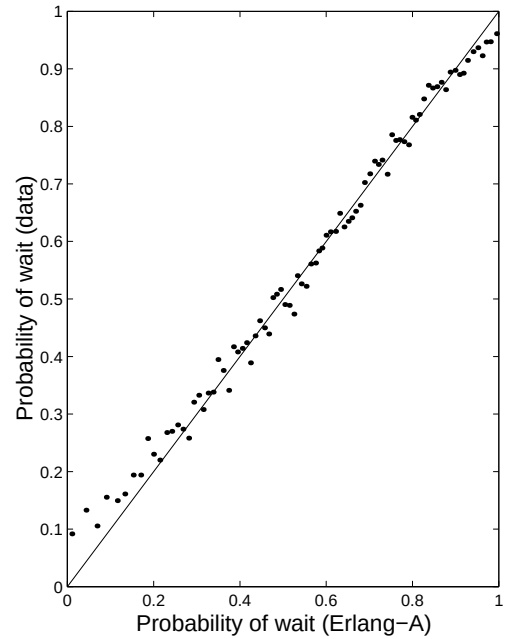
P{Ab}



E[W_q]



P{$W_q > 0$}



Erlang-A: Simple, but Not Too Simple

Further Natural Questions:

1. Why does Erlang-A practically work? justify **robustness**.
2. When does it fail? chart **boundaries**.
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Answers via **Asymptotic Analysis**, as load- and staffing-levels increase, which reveals model-essentials:

- **E**fficiency-**D**riven (**ED**) regime: Fluid models (deterministic)
- **Q**uality- and **E**fficiency-**D**riven (**QED**): Diffusion refinements.

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Motivation: Moderate-to-large service systems (**100's - 1000's** servers), notably **call-centers**.

Results turn out **accurate** enough to also cover **10-20** servers. Important – relevant to **hospitals** (nurse-staffing: de Véricourt & Jennings, 2006), ...

Operational Regimes: Conceptual Framework

Assume: **Offered Load** $R = \frac{\lambda}{\mu}$ $(= \lambda \times \mathrm{E}[S])$ not too small.

QD Regime: $N \approx R + \delta R$ $\quad [(N - R)/R \to \delta, \text{ as } N, \lambda \uparrow \infty]$

- Essentially **no** delays: $\quad [P\{W_q > 0\} \to 0]$.

ED Regime: $N \approx R - \gamma R$

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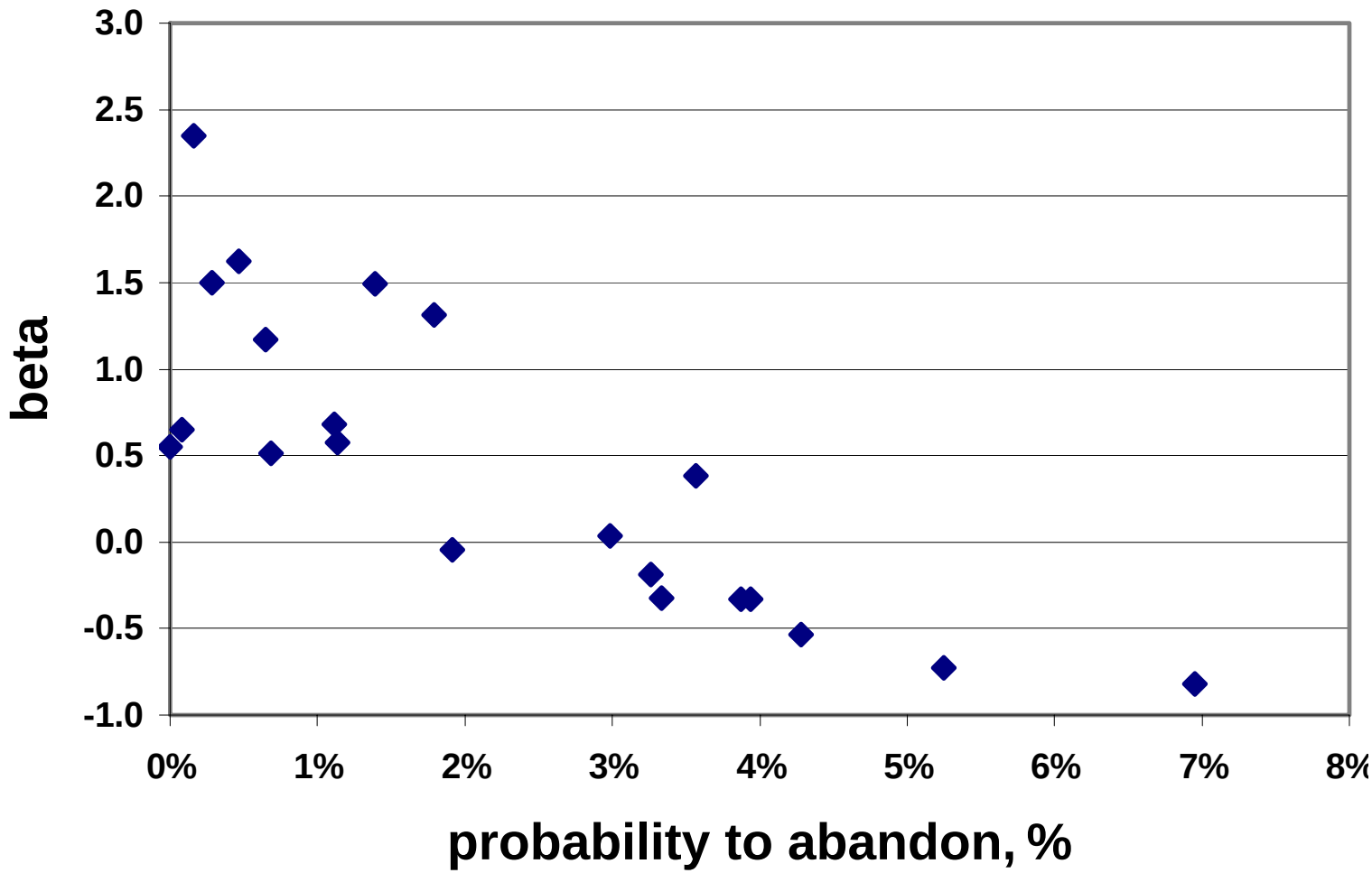
QED+ED: $N \approx (1 - \gamma)R + \beta\sqrt{R}$

- Zeltyn & M. 2006
- QED refining ED to accommodate "timely-delays": $P\{W_q > T\}$.

QED: Practical Support

QOS parameter $\beta = (N - R)/\sqrt{R}$ vs. %Abandonment



QED Theory (Erlang '13; Halfin-Whitt '81; Garnett MSc; Zeltyn PhD)

Consider a sequence of M/M/**N**+G models, **N=1,2,3,…**

Then the following **points of view** are equivalent:

- **QED** $\%\{\text{Wait} > 0\} \approx \alpha, \qquad 0 < \alpha < 1;$
- **Customers** $\%\{\text{Abandon}\} \approx \frac{\gamma}{\sqrt{N}}, \qquad 0 < \gamma;$
- Agents $\text{OCC} \approx 1 - \frac{\beta + \gamma}{\sqrt{N}} \qquad -\infty < \beta < \infty;$
- **Managers** $N \approx R + \beta\sqrt{R}$, $R = \lambda \times \text{E(S)}$ not small;

QED performance (ASA, ...) is easily computable, all in terms of β (the square-root safety staffing level) – see later.

QED Approximations (Zeltyn, M. '06)

G – patience distribution,

g_0 – patience density at origin $\quad (g_0 = \theta, \text{ if } \exp(\theta))$.

$$N = \frac{\lambda}{\mu} + \beta\sqrt{\frac{\lambda}{\mu}} + o(\sqrt{\lambda}), \qquad -\infty < \beta < \infty .$$

$$\mathsf{P}\{\mathsf{Ab}\} \approx \frac{1}{\sqrt{N}} \cdot \left[h(\hat{\beta}) - \hat{\beta}\right] \cdot \left[\sqrt{\frac{\mu}{g_0}} + \frac{h(\hat{\beta})}{h(-\beta)}\right]^{-1},$$

$$\mathsf{P}\left\{W > \frac{T}{\sqrt{N}}\right\} \approx \left[1 + \sqrt{\frac{g_0}{\mu}} \cdot \frac{h(\hat{\beta})}{h(-\beta)}\right]^{-1} \cdot \frac{\bar{\Phi}\left(\hat{\beta} + \sqrt{g_0\mu} \cdot T\right)}{\bar{\Phi}(\hat{\beta})},$$

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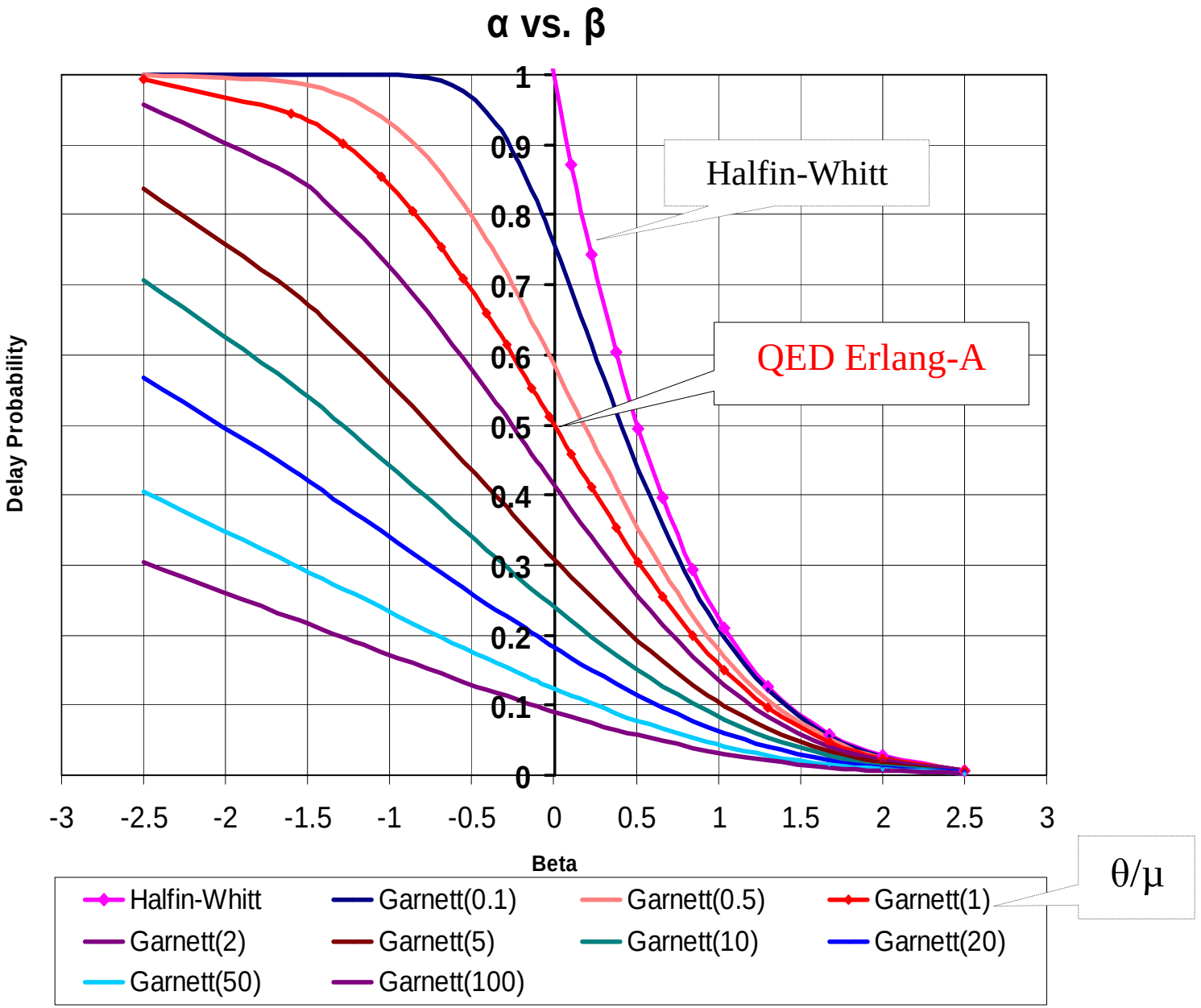
Here

$$\hat{\beta} = \beta\sqrt{\frac{\mu}{g_0}}$$

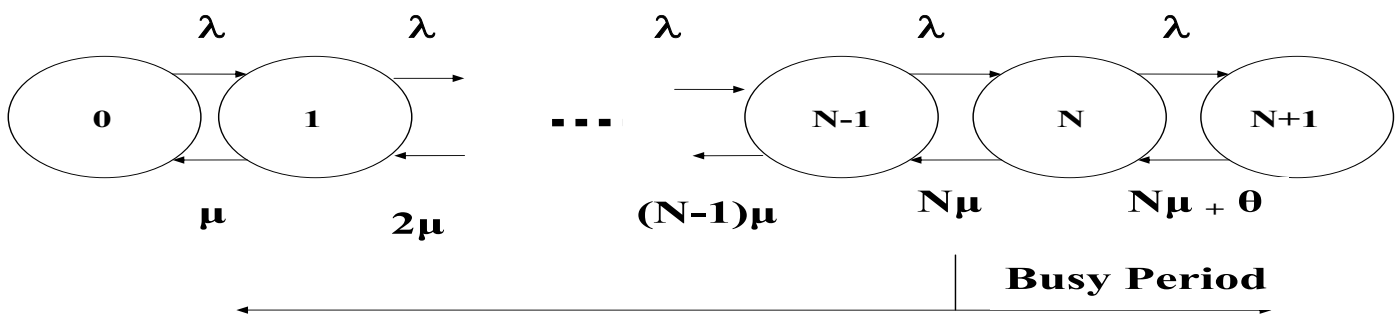
$$\bar{\Phi}(x) = 1 - \Phi(x),$$

$$h(x) = \phi(x)/\bar{\Phi}(x), \text{ hazard rate of } N(0,1).$$

Garnett / Halfin-Whitt Functions: $P\{W_q > 0\}$



QED Intuition via Excursions: Busy/Idle Periods



$Q(0) = N$: all servers busy, no queue.

Let $T_{N,N-1}$ = Busy Period (down-crossing $N \downarrow N-1$)

$T_{N-1,N}$ = Idle Period (up-crossing $N-1 \uparrow N$)

Then $$P(Wait > 0) = \frac{T_{N,N-1}}{T_{N,N-1} + T_{N-1,N}} = \left[1 + \frac{T_{N-1,N}}{T_{N,N-1}}\right]^{-1}$$

QED Intuition via Excursions: Asymptotics

Calculate $$T_{N-1,N} = \frac{1}{\lambda_N E_{1,N-1}} \sim \frac{1}{N\mu \times h(-\beta)/\sqrt{N}} \sim \frac{1}{\sqrt{N}} \cdot \frac{1/\mu}{h(-\beta)}$$

$$T_{N,N-1} = \frac{1}{N\mu\pi_+(0)} \sim \frac{1}{\sqrt{N}} \cdot \frac{\beta/\mu}{h(\delta)/\delta}, \quad \delta = \beta\sqrt{\mu/\theta}$$

Both apply as $\sqrt{N}\,(1-\rho_N) \to \beta,\ -\infty < \beta < \infty.$

Hence, $$P(Wait > 0) \sim \left[1 + \frac{h(\delta)/\delta}{h(-\beta)/\beta}\right]^{-1}.$$

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Special case: $\mu = \theta$ (Impatient):

Then $\boldsymbol{Q \stackrel{d}{=} M/M/\infty}$, since sojourn-time is exp($\mu = \theta$).

If also $\beta = 0$ (Prevalent): $\boldsymbol{P\{Wait > 0\} \approx 1/2}$.

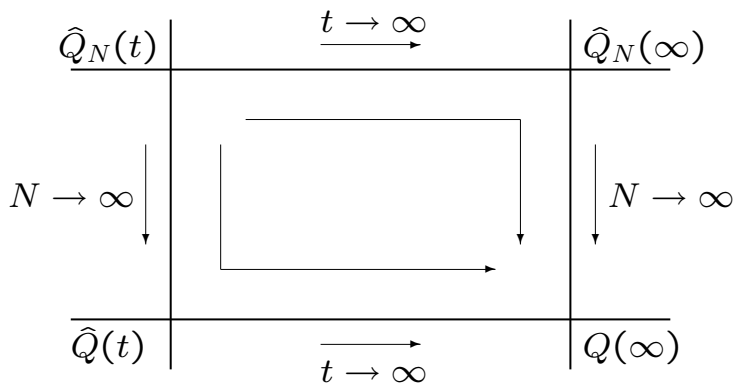
Process Limits (Queueing, Waiting)

- $\hat{Q}_N = \{\hat{Q}_N(t), t \geq 0\}$: stochastic process obtained by centering and rescaling:

$$\hat{Q}_N = \frac{Q_N - N}{\sqrt{N}}$$

- $\hat{Q}_N(\infty)$: stationary distribution of $\hat{Q}_N$

- $\hat{Q} = \{\hat{Q}(t), t \geq 0\}$: process defined by: $\hat{Q}_N(t) \xrightarrow{d} \hat{Q}(t)$.



Approximating (Virtual) Waiting Time

$$\hat{V}_N = \sqrt{N}\, V_N \Rightarrow \hat{V} = \left[\frac{1}{\mu}\, \hat{Q}\right]^+$$ (Puhalskii, 1994)

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Operational Regimes provide a **conceptual framework**.

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Questions:

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   - **Constraint Satisfaction**: eg. **Min. N , s.t. QOS constraints** .

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   - **Constraint Satisfaction**: eg. **Min. N , s.t. QOS constraints** .
3. Robustness depends:
   - **Without Abandonment: QED** covers all, at amazing accuracy.
   - **With Abandonment: ED, QED, ED+QED** all have a role.

Operational Regimes: Rules-of-Thumb

Constraint	P{Ab}		E[W]		P{$W > T$}	
Offered Load	Tight 1-10%	Loose $\geq 10\%$	Tight $\leq 10\%\mathrm{E}[\tau]$	Loose $\geq 10\%\mathrm{E}[\tau]$	Tight $0 \leq T \leq 10\%\mathrm{E}[\tau]$ $5\% \leq \alpha \leq 50\%$	Loose $T \geq 10\%\mathrm{E}[\tau]$ $5\% \leq \alpha \leq 50\%$
Small (10's)	QED	QED	QED	QED	QED	QED
Moderate-to-Large (100's-1000's)	QED	ED, QED	QED	ED, QED if $\tau \stackrel{d}{=} \exp$	QED	ED+QED

ED: $N \approx R - \gamma R \quad (0.1 \leq \gamma \leq 0.25\,)$.

QD: $N \approx R + \delta R \quad (0.1 \leq \delta \leq 0.25\,)$.

QED: $N \approx R + \beta\sqrt{R} \quad (-1 \leq \beta \leq 1\,)$.

ED+QED: $N \approx (1-\gamma)R + \beta\sqrt{R} \quad (\gamma, \beta$ as above).

Back to “Why does Erlang-A Work?"

Theoretical Answer: $M_t^J/G/N_t + G \overset{d}{\approx} (M/M/N+M)_t\,, \quad t \geq 0.$

- **General Patience**: Behavior at the origin is all that matters.
- **General Services**: Empirical insensitivity beyond the mean.
- **Time-Varying Arrivals**: Modified Offered-Load approximations.
- **Heterogeneous Customers**: 1-D state collapse.

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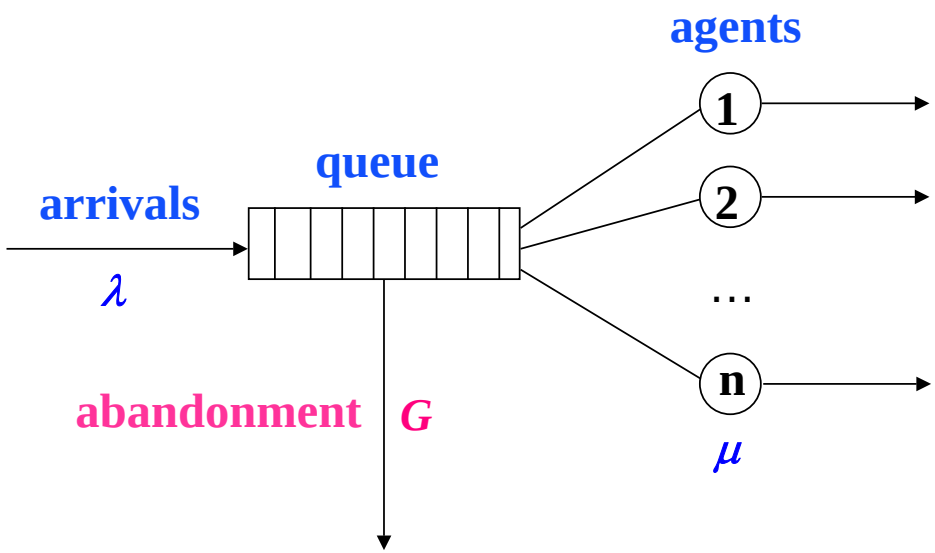
- **General Services**: Empirical insensitivity beyond the mean.
- **Time-Varying Arrivals**: Modified Offered-Load approximations.
- **Heterogeneous Customers**: 1-D state collapse.

Practically: Why do (stochastic-ignorant) Call Centers work?

"The right answer for the wrong reason"

"Why does Erlang-A Work?" General Patience



(Im)Patience times **Generally Distributed**: M/M/***n***+G

Exact analysis in steady-state (Baccelli & Hebuterne, 1981): solve Kolmogorov's PDE's (semi-Markov) for the offered-wait ***V***; Generalized by Brandt & Brandt in late 90's.

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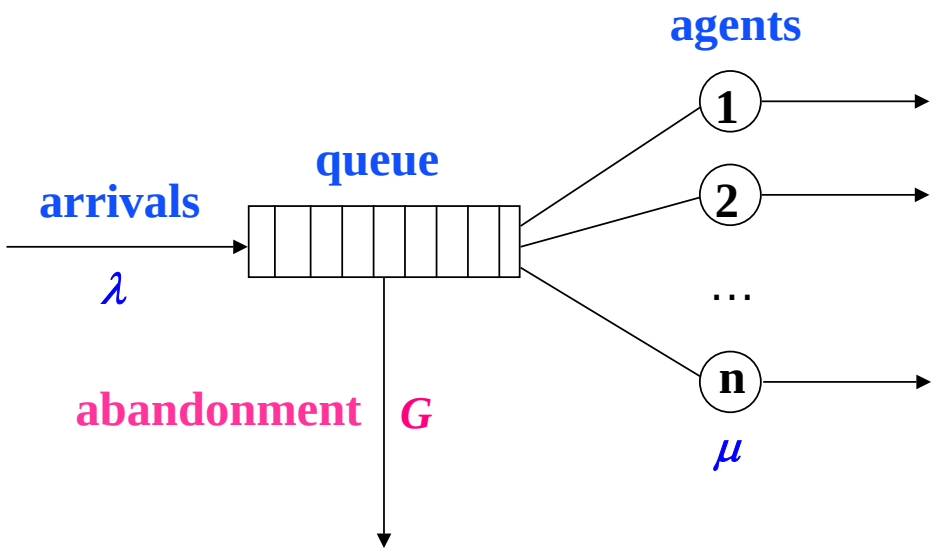

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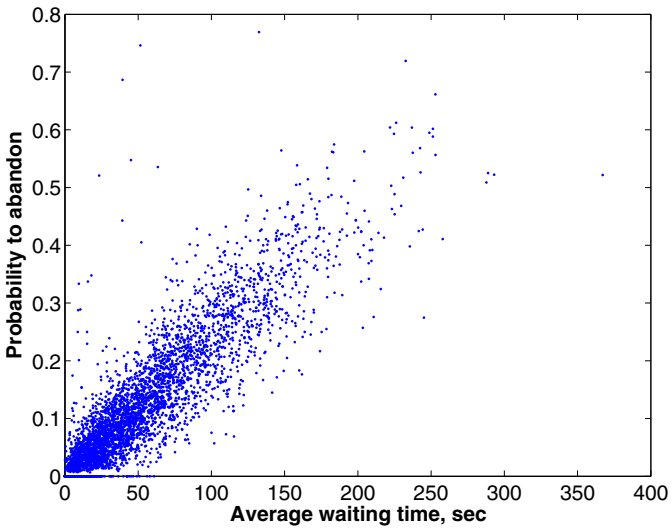
QED analysis (w/ Zeltyn, 2006): $\boldsymbol{n} \approx \boldsymbol{R} + \beta\sqrt{\boldsymbol{R}}$.

- Assume (Im)Patience density $\boldsymbol{g(0)} > \boldsymbol{0}$.
- $\boldsymbol{V}$ asymptotics ($\lambda \uparrow \infty$): Laplace Method, leading to
- **QED Approximations: Use Erlang-A** as is, with $\theta \leftrightarrow \boldsymbol{g(0)}$.

General Patience: Fitting Erlang-A

Israeli Bank: Yearly Data

Hourly Data



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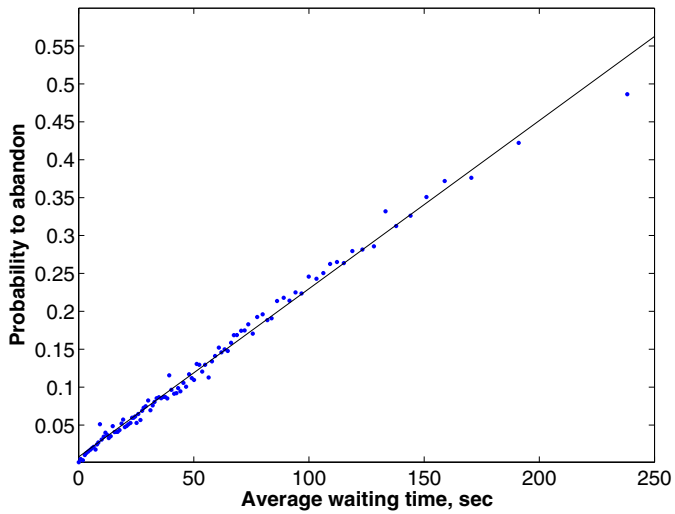

Theory:

Erlang-A: $\mathrm{P}\{\mathrm{Ab}\} = \theta \cdot \mathrm{E}[W_q]$;

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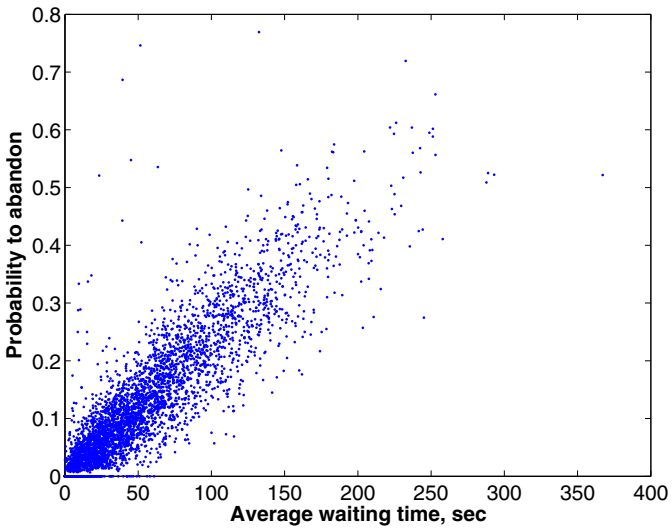

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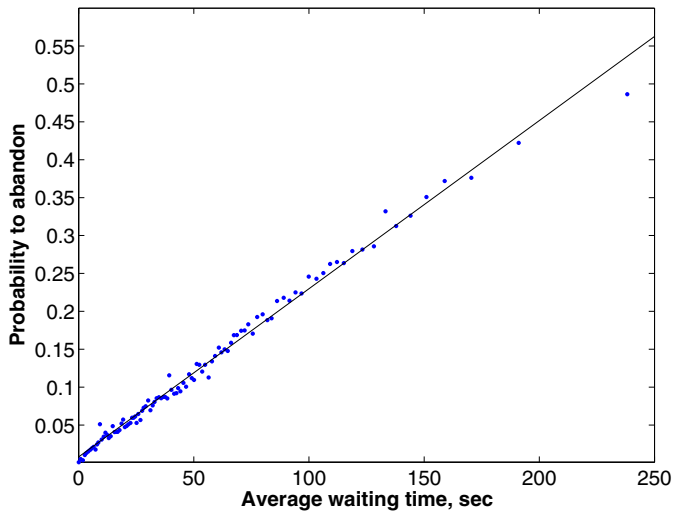

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Recipe:

In both cases, use Erlang-A, with $\hat{\theta} = \widehat{\mathrm{P\{Ab\}}}/\widehat{\mathrm{E}[W_q]}$ (slope above).

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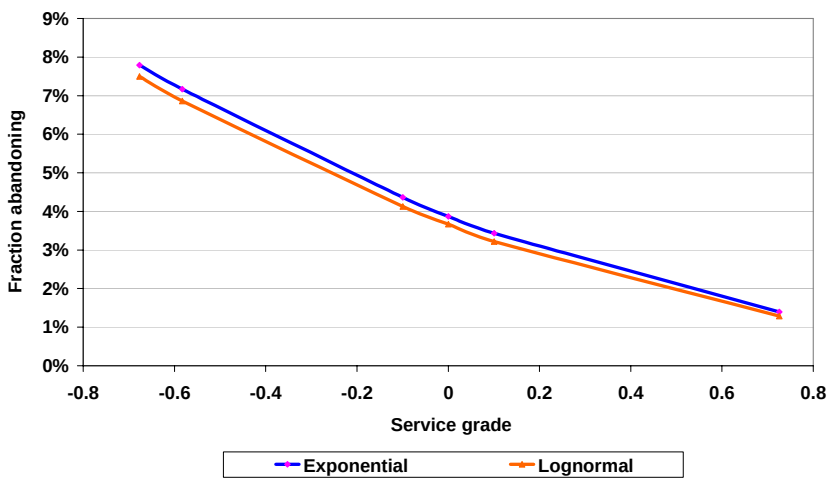
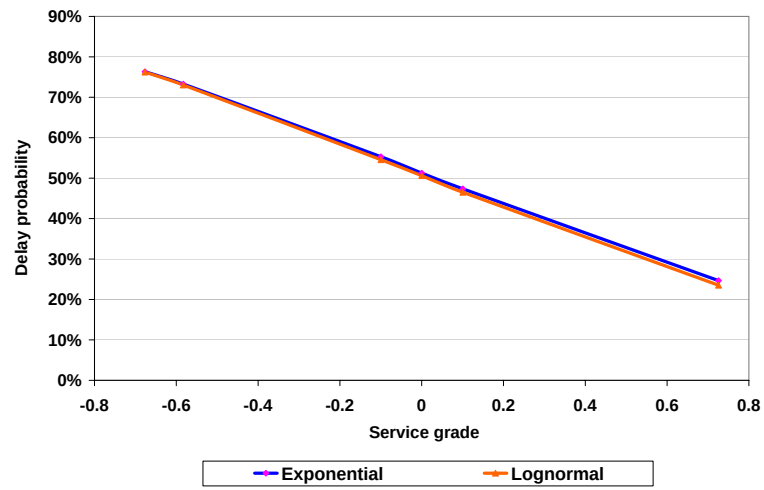
Fraction Abandoning



Delay Probability



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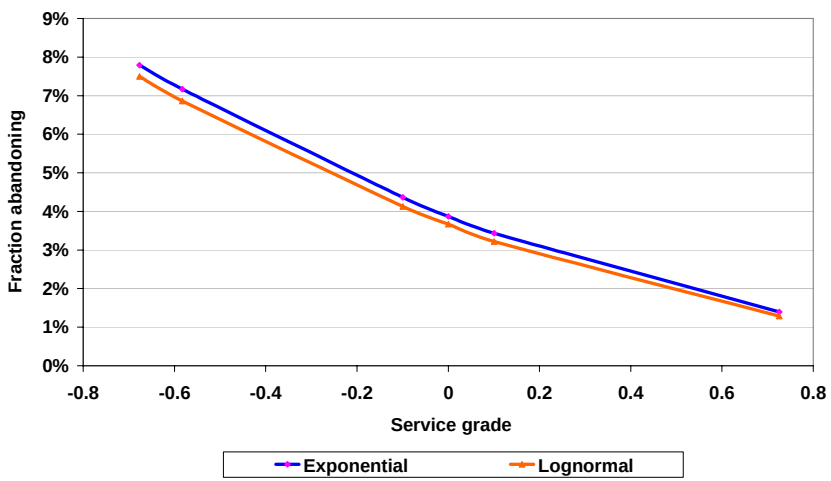

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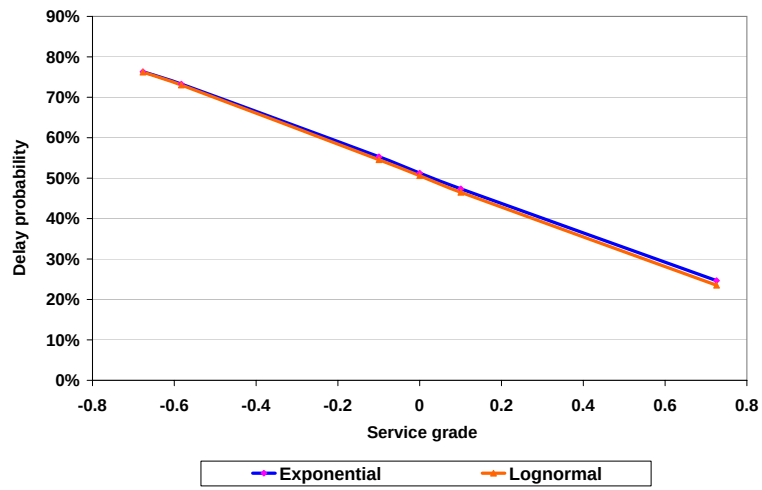

QED G-Services: G/D_K/N+G (w/ Momčilović, ongoing).

Why Does Erlang-A Work? Time-Varying Arrival Rates

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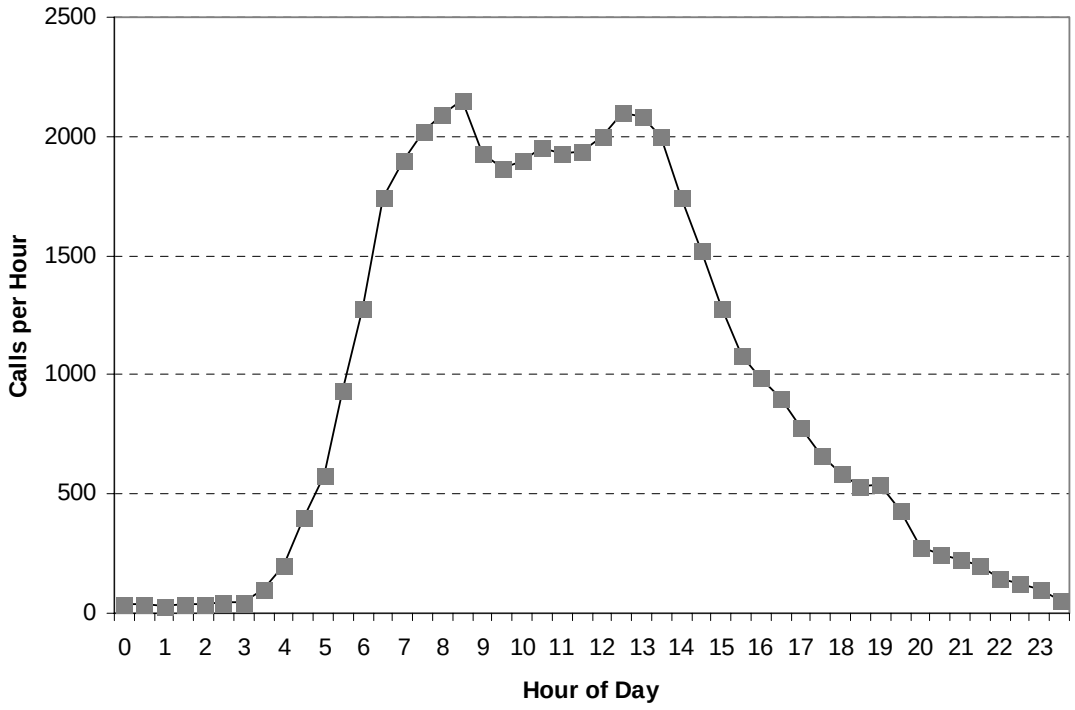
Serendipity: **Time-stable** performance, supported by **ISA** = Iterative Staffing Algorithm, and QED diffusion limits ($M_t/M/N + M, \ \mu = \theta$).

Example: "Real" Call Center

(The "Right Answer" for the "Wrong Reasons")

Time-Varying (two-hump) arrival functions common

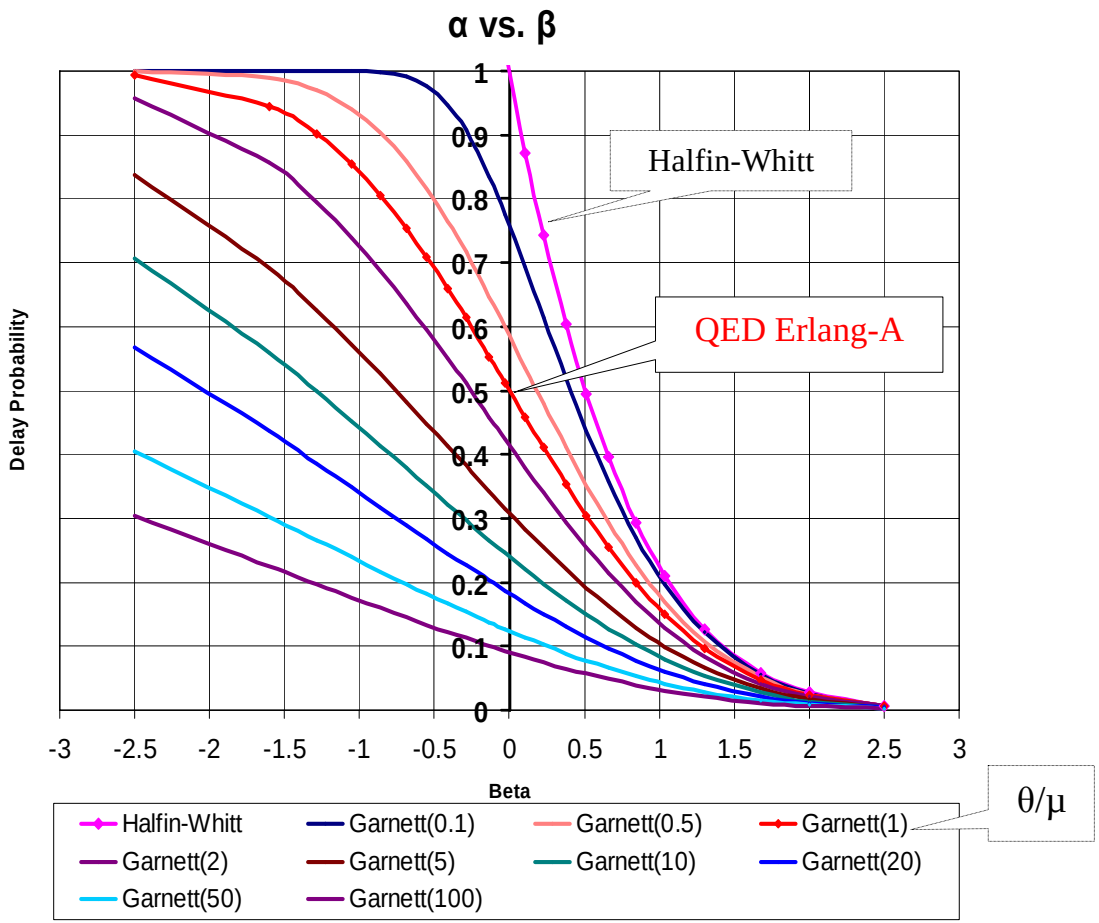
(Adapted from Green L., Kolesar P., Soares J. for benchmarking.)



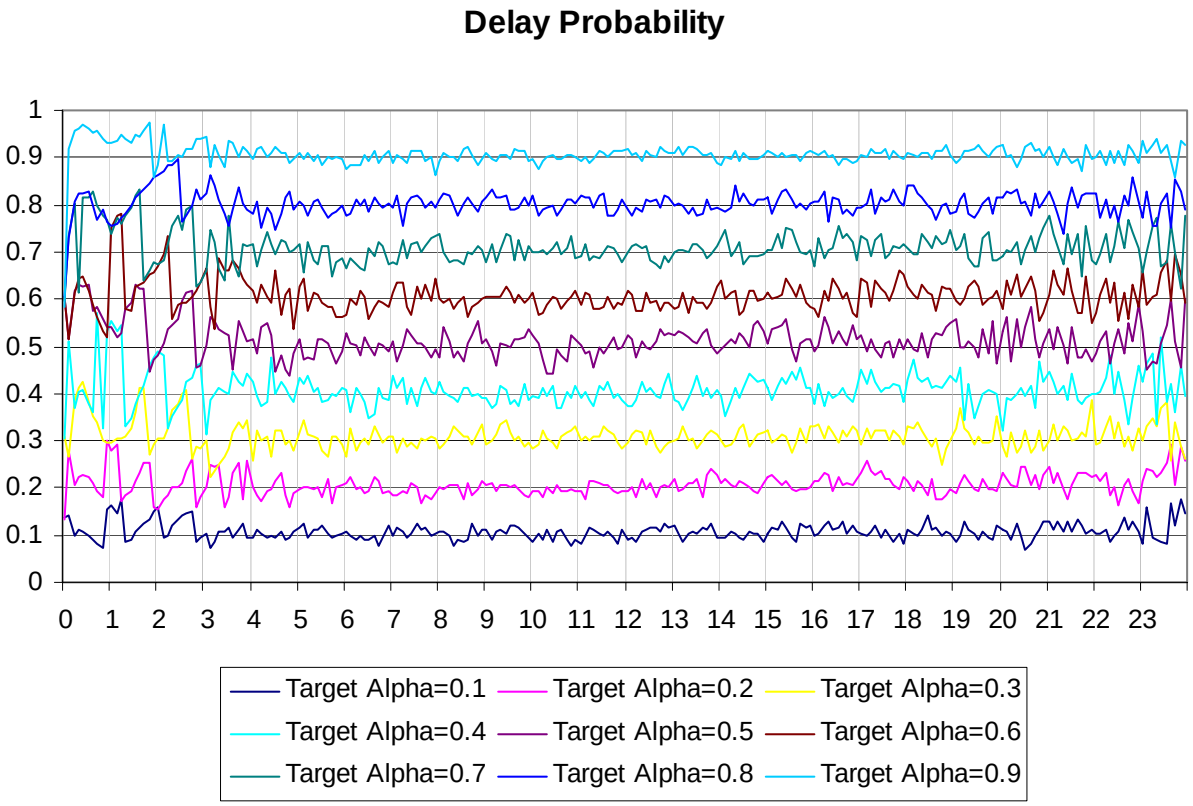
Assume: Service and abandonment times are both

Exponential, with mean 0.1 (6 min.)

HW/GMR Delay Functions

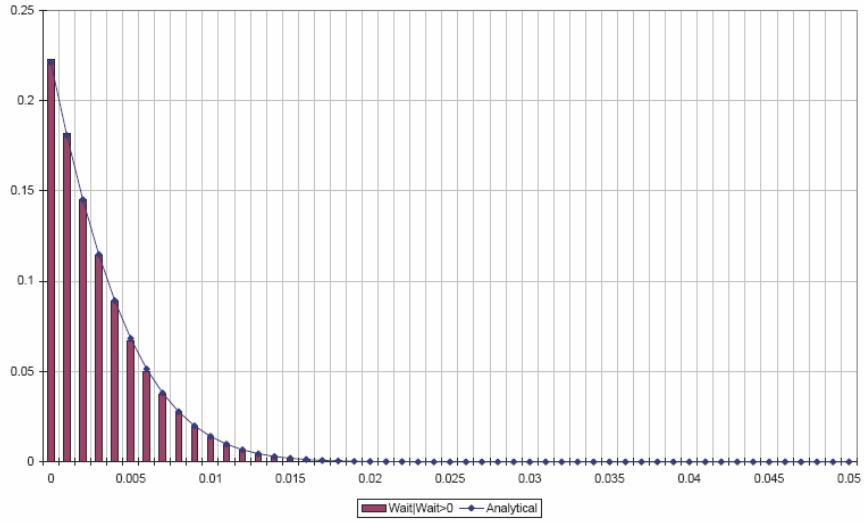


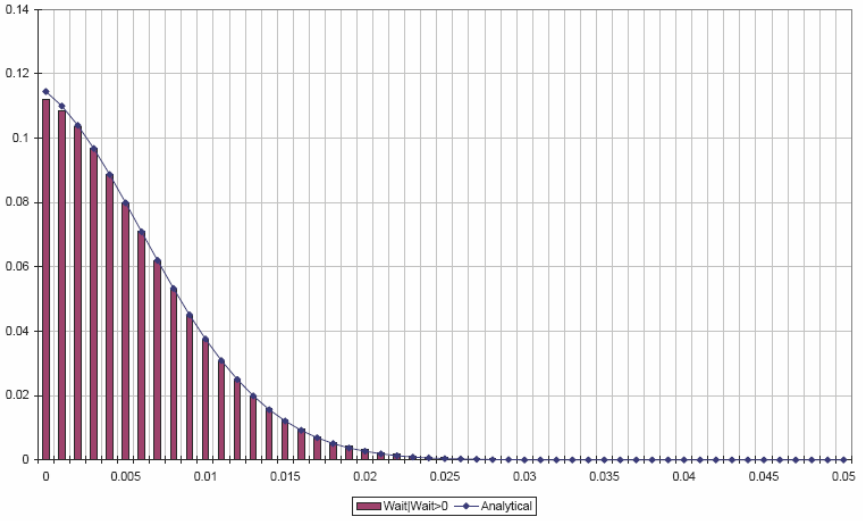
Delay Probability α

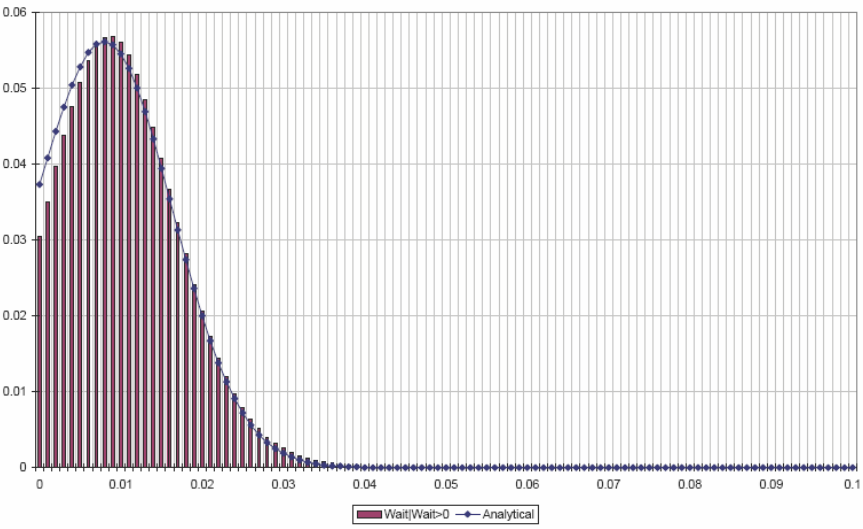


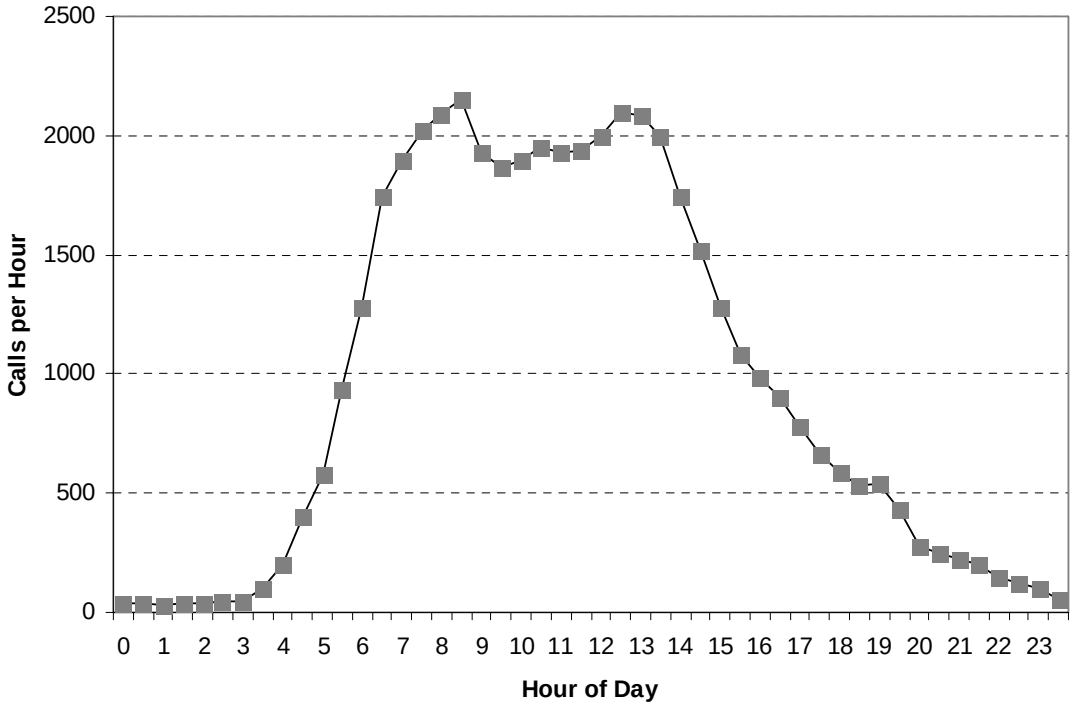
Real Call Center: Empirical waiting time, given positive wait

(1) α=0.1 (*QD*) **(2)** α=0.5 (*QED*) **(3)** α=0.9 (*ED*)

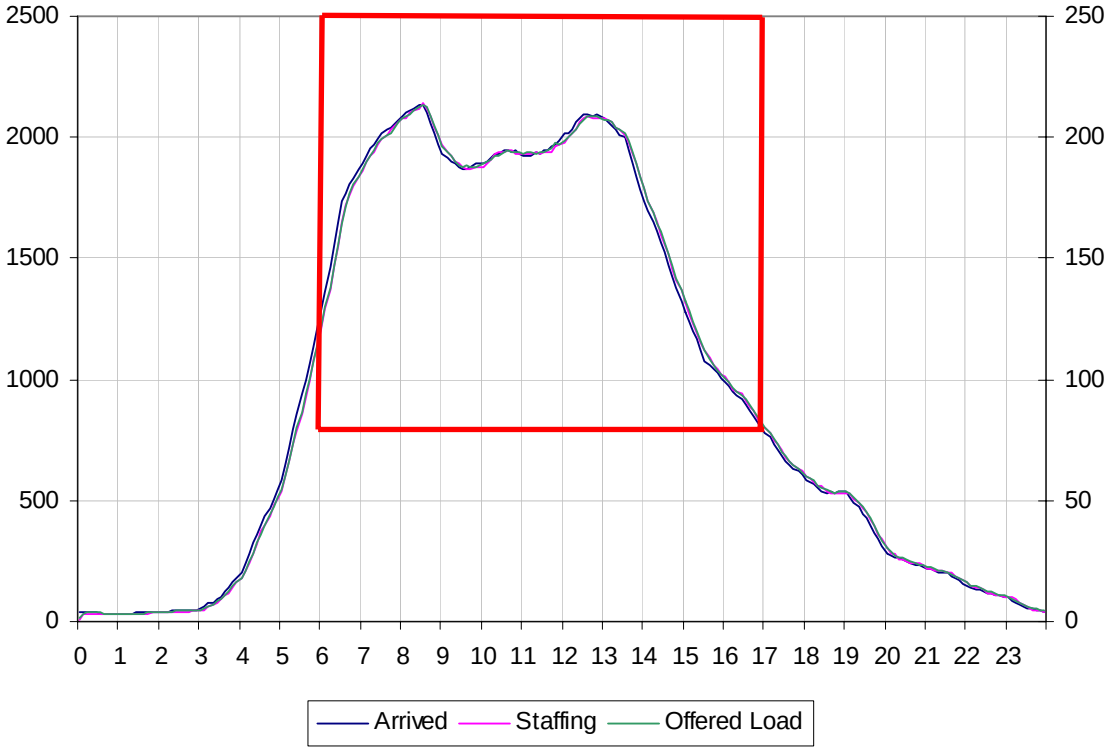








QED Staffing (β=0 iff α=0.5)



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Now: $M_t^J/G/N_t + G \approx (M^J/G/N + G)_t$ (well staffed & controlled).

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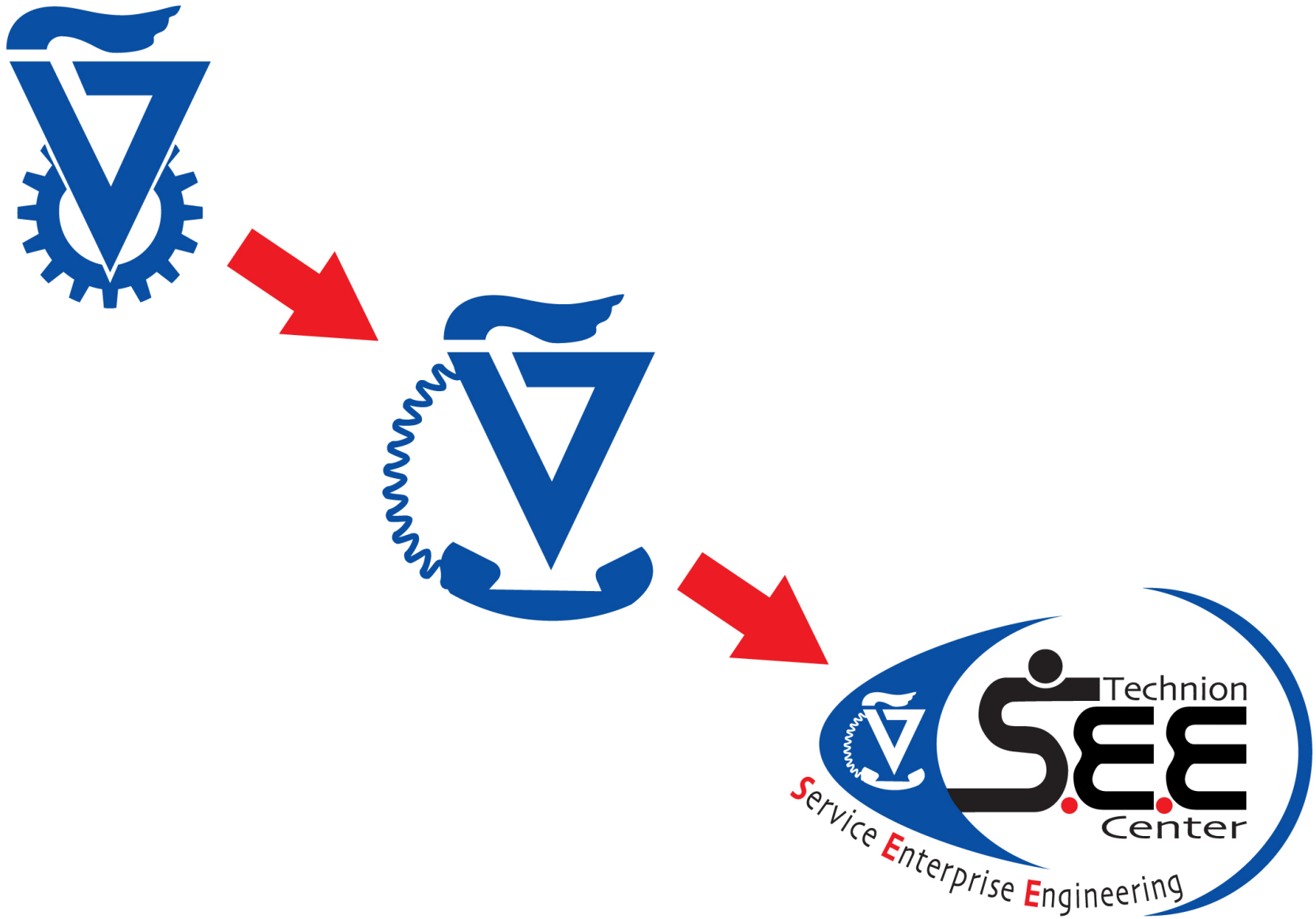
Serendipity: Multi-Class Multi-Skill, w/ **class-dependent** services.
Support: ISA, QED diffusion limits (Atar, M. & Shaikhet, 2007).

The Technion SEE Center / Laboratory



DataMOCCA = Data MOdels for Call Center Analysis

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- **Wharton**: L. Brown, N. Gans, H. Shen (UNC), Zhao.
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Free for academic adoption: **ask for a DVD (3GB)**.