LEONTIEF SYSTEMS, RBV'S and RBM'S.

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Abstract A stochastic continuous-time input-output model is proposed, which is an analog of the classical Leontief system in Economics. Feasibility and efficiency results are established in terms of a dynamic version of the Linear Complementarity Problem from Mathematical Programming. Solving this dynamic version amounts to subjecting a given process to oblique reflection on an orthant or a simplex. Explicitly considered are Reflected processes of Bounded Variation (RBV's), and Reflected Brownian Motions (RBM's). It is explained how RBM's arise as diffusion approximations to Leontief systems, when the latter are heavily-loaded and driven by RBV's.

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1. Introduction

The current work is concerned with some mathematical problems, stimulated by models of flow networks and Leontief economies. Flow networks are analyzed in Chen and Mandelbaum [1989a, 1989b] (abbreviated to CMa and CMb in the sequel). The subject here is Leontief economies; specifically, representations and approximations of dynamic, stochastic, continuous-time analogs of the classical input-output Leontief systems in economics. (The classical models are described, for example, in Gale [1960], Nikaido [1968], Berman and Plemmons [1979] or Veinott [1968,1969].) We focus on mathematical foundations, while leaving potentially interesting economic interpretations and applications as a future challenge.

Two particular cases of our stochastic model have been thoroughly studied. The first case constitutes one-dimensional systems; their driving source of uncertainty are Levy processes whose sample paths are locally of bounded variation. The resulting models are often called (generalized) storage models. They are covered in Moran [1959] and Prabhu [1965,1980] (see also Kaspi [1984] for an extension to models driven by Markov additive processes, and consult the survey by Bingham [1975] for further references). Our Leontief models, as well as their flow-networks counterpart, can all be thought of as *multi-dimensional* generalizations of the one-dimensional storage models. These generalizations turn out to have a geometric interpretation, briefly described in 2.5.C below, which justifies us calling them Reflected, or Regulated, processes of Bounded Variation (RBV's for short).

The terminology "RBV" also stems from the name attached to the second thoroughly studied case. These are models with sources of uncertainty that are multi-dimensional Brownian motions, and the geometric interpretation 2.5.C inspired calling them Reflected, or Regulated, Brownian Motions (RBM's). Key references here are Skorohod [1961] (the original one-dimensional model), Tanaka [1979] (a multi-dimensional RBM with normal reflection), Harrison and Reiman [1981] (which is the source for the models considered here), Harrison [1985] (especially Chapter 2, a recommended prerequisite for this paper), and Harrison and Williams [1987a] and Williams [1987] (who introduce RBM's with state spaces that are general polyhedral domains).

In our paper, readers will encounter RBM's whose state space is either a nonnegative orthant or a simplex. Orthants are associated with *open* RBM's (Harrison and Williams [1987b]) and simplices with

closed RBM's (Harrison, Williams and Chen [1989]). Open RBM's arose as diffusion approximations for balanced *open* queueing networks in Reiman [1984], and were formally introduced in Harrison and Reiman [1981]. Closed RBM's arose as diffusion approximations for congested *closed* queueing networks in CMb, and the need to develop for them an analog of Harrison and Reiman [1981] then emerged. This was our primary goal (and it is formally achieved via Theorems 2.6 and 2.7). But it has also turned out that sources of uncertainty which are not Brownian (for example, point processes or subordinators,) give rise to processes interesting in their own right (for example, RBV's). These processes share, furthermore, a *direct* physical interpretation that is lacking in Brownian models. One such interpretation was provided by the fluid networks in CMa. An alternative interpretation is introduced here in terms of Leontief input-output systems.

In Section 2 we define a *deterministic* Leontief system in continuous time (or equivalently, a sample path of a *stochastic* Leontief system). We then focus, in Section 3, on closed models, specifically on closed RBV's and closed RBM's. As an illustrative exercise, it is shown that closed RBM's provide diffusion approximations for heavily congested closed RBV's. (The exercise serves as an introductory glimpse to the way RBM's emerge as diffusion approximations for nonparametric Jackson queueing networks, as in Reiman [1984] and CMb). Section 4 consists of proofs for most of the results stated in Section 2. The paper ends, in Section 5, with closing remarks and some possibly feasible directions for future research.

Before proceeding, readers are advised to skim through the Appendix; it lists notation and conventions that are used in the sequel.

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2. A Deterministic Leontief System

For our purposes, a Leontief economy (Nikaido [1968], §6.2) is a collection of J production sectors, each producing a single commodity. We shall identify a sector with the commodity it produces, labeling commodities by either i or j, $i, j = 1, \ldots, J$. The production of each single unit of commodity j directly consumes $q_{ij} \geq 0$ units of each commodity i. There is given a ***netput*** process $x = \{x(t), t \geq 0\}$, $x \in D_0^J$, whose i-th coordinate at time t, $x_i(t)$, represents the net of *cumulative exogenous* amount of supply (input) less demand (output) of commodity i, ***accumulated*** over the time interval $[0, t]$.

2.1 A possible temporal evolution of the economy is modeled by a pair (y, z), $y \in D_\uparrow^J$, $z \in D_+^J$, which satisfies, for $t \geq 0$ and $i = 1, \ldots, J$, the flow balance equations

2.1.A $z_i(t) = x_i(t) + y_i(t) - \sum_{j=1}^J q_{ij} y_j(t)$, as well as

2.1.B $y_i(t)$, and hence $z_i(t)$, depends only on the restriction of x to $[0, t]$.

Here $y_i(t)$ and $z_i(t)$ are, respectively, the i-th coordinates of y and z at time $t \geq 0$: $y_i(t)$ is interpreted as a feasible *cumulative* production amount of commodity i during the time interval $(0, t]$, hence $z_i(t)$ describes the work-in-process inventory of commodity i at time t. (The monotonicity of y excludes perishability of products, and the nonnegativity of z excludes backlogs). Property 2.1.B guarantees that the recipe $y(t)$ for production at each present time $t \geq 0$, does not involve future data that is revealed by $\{x(s), s > t\}$. Formally, y is to be viewed as a function $y(x)$, $x \in D_0^J$, such that $y(x^1)$ and $y(x^2)$ coincide on $[0, t]$ whenever x^1 and x^2 do, for all $t \geq 0$.

2.1.C *Remark:* With the interpretation given to "netput", the process x ought to be a difference between two nondecreasing functions, hence of bounded variation over every finite interval (locally of bounded variation). The bounded variation feature, however, will *rarely* be used in Section 4, and for a good reason: to cover RBM's, in particular the diffusion approximations in Section 3, requires that our theory accomodates netputs which are sample paths of a Brownian motion; these, of course, are of unbounded variation over every finite interval.

Call the nonnegative $J \times J$ matrix $Q = [q_{ij}]$ a ***consumption*** matrix. We shall find it convenient to use 2.1.A in its vector form

2.1.D $z = x + [I - Q]y$,

and paraphrase 2.1.B as

2.1.E y, and hence z, are non-anticipating with respect to x.

The process $y \in D_\uparrow^J$ will be called a feasible ***production plan*** responding to the netput x, and $z = x + [I - Q]y \in D_+^J$ will be referred to as the ***inventory*** process associated with y.

2.2 Every nonnegative matrix Q can play the role of a consumption matrix. For a given $Q \geq 0$, a production plan responding to the netput $x \in D_0^J$ is an element of the ***production set*** $\pi_Q(x)$ defined by

$$\pi_Q(x) = \{y \in D_\uparrow^J : x + [I - Q]y \geq 0\} . \tag{2.1}$$

We are interested in an efficient production plan, in the sense that it produces the least required to respond to a given netput.

Proposition 2.1. *A non-empty production set has a unique least production plan. Formally, $\pi_Q(x) \neq \emptyset$ implies the existence of a unique $\underline{y} \in \pi_Q(x)$ such that*

$$\underline{y}_j(t) \leq y_j(t) \quad , \quad \textit{for all } j = 1, \ldots, J \ , \ t \geq 0 \ ,$$

whenever $y \in \pi_Q(x)$. This $\underline{y}$ is the lower envelope of $\pi_Q(x)$, and it is non-anticipating with respect to x.

Proof: It will be shown that $\underline{y} = \{(\underline{y}_1(t), \ldots, \underline{y}_J(t))', t \geq 0\}$, defined for $j = 1, \ldots, J$ and $t \geq 0$ by

$$\underline{y}_j(t) = \inf\{y_j(t) : y \in \pi_Q(x)\} \ , \tag{2.2}$$

is a least production plan in $\pi_Q(x)$. To this end, one must check that $\underline{y} \in \pi_Q(x)$ and $\underline{y}$ is non-anticipating with respect to x. Clearly $\underline{y}$ is non-decreasing with $\underline{y}(0) = 0$. The fact that $\underline{y} \in D_\uparrow^J$ is a consequence of

2.2.A Let $\{f_\alpha\}$ be a sub-family in $D_\uparrow^1$. Then its lower envelope $\underline{f}$, given by

$$\underline{f}(t) = \inf_\alpha f_\alpha(t) \quad , \quad t \geq 0 \ ,$$

is also in $D_\uparrow^1$

(One way of verifying 2.2.A is to observe that for a real-valued non-decreasing function on $[0, \infty)$, right continuity is equivalent to upper-semi-continuity, and the lower envelope of upper-semi-continuous functions is also upper-semi-continuous; see Ash [1972], page 389.) Now take $\epsilon > 0$ and fix i and t. There exists $y \in \pi_Q(x)$ such that $y_i(t) \le \underline{y}_i(t) + \epsilon$ and $y_j(t) \ge \underline{y}_j(t)$ for all $j = 1, \ldots, J$. It follows that

$$x_i(t)+\underline{y}_i(t)-\sum_{j=1}^{J} q_{ij}\underline{y}_j(t) \ge x_i(t)+y_i(t)-\sum_{j=1}^{J} q_{ij}y_j(t)-\epsilon \ge -\epsilon\,. \quad (2.3)$$

Since $\epsilon > 0$ in (2.3) is arbitrary, $x + [I - Q]\underline{y} \ge 0$. To establish non-anticipativity, for $t > 0$ let

$$\pi_Q^t(x) = \{y \in D_\uparrow^J : x(s) + [I - Q]y(s) \ge 0 \;\; for \;\; 0 \le s \le t\}\,,$$

and define for $j = 1, \ldots, J$ and $s \in [0, t]$

$$\underline{y}_j^t(s) = \inf\{y_j(s) : y \in \pi_Q^t(x)\}\,.$$

Let us show that the restriction of $\underline{y}$ to $[0, t]$, to be denoted by $\underline{y}[0, t]$, coincides with $\underline{y}^t$. Indeed, from $\underline{y} \in \pi_Q^t(x)$ first follows that $\underline{y}^t \le \underline{y}[0, t]$. Now producing according to $\underline{y}^t$ on $[0, t)$, followed by $\underline{y}$ on $[t, \infty)$, provides a feasible production plan. Therefore, (2.2) implies that $\underline{y}^t \ge \underline{y}[0, t]$, which completes the proof.

The least production plan $\underline{y}$, identified in Proposition 2.1, will be called an *efficient* production plan. It was introduced in Reiman [1984] (Proposition 1 in the Appendix), where it is shown that $\pi_Q(x)$ is a meet semi-lattice that is bounded below. We add completeness to this semi-lattice, and to Reiman's argument as well.

2.3 An element $y \in \pi_Q(x)$, $x \in D_0^J$, is ***complementary*** to $z = x + [I - Q]y$ if each y_j increases at $t \ge 0$ only when $z_j(t) = 0$, $j = 1, \ldots, J$. The interpretation is that plan y calls for producing commodity j only when there is none of it in stock. Equivalently,

2.3.A $\quad \int_0^\infty z\,dy = 0\,,$

where the integral stands for the sum of the J zero-valued Lebesgue-Stieltjes integrals $\int_0^\infty z_j(t)\,dy_j(t)$, $j = 1, \ldots, J$. We shall, alternatively, refer to $y \in \pi_Q(x)$ satisfying 2.3.A as a ***regulator*** of $x \in D_0^J$, and to the pair (y, z) as a solution to the ***Dynamic Complementarity Problem*** with data x and matrix Q (abbreviated to $DCP_Q(x)$, and sometimes only DCP, for covenience).

DCP is a dynamic continuous-time version of the Linear Complementarity Problem (LCP) from mathematical programming (Cottle and Dantzig [1968], Mandelbaum [1989]). The relation between LCP and DCP is revealed, in Subsection 4.1, through specializing $DCP_Q(x)$ to netputs x that change values in isolated jumps only. (A recent comprehensive coverage of LCP is Murty [1988].)

In view of the interpretation given to complementarity, the following should come at no surprise:

Proposition 2.2. *An efficient production plan in $\pi_Q(x)$ provides a solution to $DCP_Q(x)$. Formally, let $Q \ge 0$, and suppose that $\pi_Q(x) \ne \emptyset$ for some $x \in D_0^J$. Then $(\underline{y}, \underline{z})$ solves $DCP_Q(x)$, where $\underline{y}$ is the least element of $\pi_Q(x)$, and $\underline{z} = x + [I - Q]\underline{y}$.*

Proposition 2.2 is proved in Subsection 4.2. Its converse need not hold. For example, let Q be an irreducible column-stochastic matrix, and take $x(t) \equiv 0$, $t \ge 0$. Clearly, the unique least element in $\pi_Q(x)$ is $\underline{y} \equiv 0$. But let ξ be a Peron-Fröbenius right-eigenvector of Q, namely $\xi > 0$ with $Q\xi = \xi$. Then for any two positive numbers α, β, the process defined to be zero on $[0, \alpha)$ and $\beta \cdot \xi$ on $[\alpha, \infty)$, solves DCP with data $x \equiv 0$.

2.4 Propositions 2.1 and 2.2 tell us that a regulator for an $x \in D_0^J$ must be the least element of $\pi_Q(x)$ if and only if $DCP_Q(x)$ has a unique solution. Identifying pairs x and Q for which such existence and uniqueness holds is our main goal in the remainder of the present section.

To recapitulate, solving $DCP_Q(x)$ for an $x \in D_0^J$ amounts to finding a regulator $y \in D_0^J$ of x such that

2.4.A $\quad z = x + [I - Q]y \ge 0\,,$

2.4.B $\quad y$ is non-decreasing with $y(0) = 0$, and

2.4.C $\quad y_j$ increases only at times $t \ge 0$ when $z_j(t) = 0$, $j = 1, \ldots, J$.

Consider an $x \in D_0^J$ for which a regulator y ***exists and is unique***. At such an x, we define the ***regulator mapping*** Ψ_Q by $\Psi_Q(x) = y$, and say either that x is ***in the domain*** of Ψ_Q, or that Ψ_Q is *well defined* at x. Let x belong to the domain of Ψ_Q. We call Ψ_Q ***continuous*** at x if for any sequence $\{x_n\}$ in its domain that converges *u.o.c.* to x, the sequence $\{\Psi_Q(x_n)\}$ converges *u.o.c.* to $\Psi_Q(x)$.

2.5 A Leontief economy is called *open* if the spectral radius of its consumption matrix $Q \ge 0$ is less than unity ($\sigma(Q) < 1$); it is called *closed* if $\sigma(Q) = 1$ and Q is irreducible. The case $\sigma(Q) > 1$ will be excluded from further considerations. To see why, let $\xi > 0$ be any positive vector. Define $\Lambda = diag(\xi)$, and observe that $\tilde{Q} = \Lambda Q \Lambda^{-1}$ has the same spectral radius as Q. Furthermore, y, z satisfy 2.4.A-2.4.C if and only if Λy and Λz satisfy 2.4.A-2.4.C with $\tilde{Q}$ and Λx. This transformation in terms of Λ amounts to merely changing the units in which commodities are measured. Now when $\sigma(Q) > 1$, there are ξ's for which $\xi' Q > \xi'$ (for example, Peron-Fröbenius left-eigenvectors of Q). Changing units with such a ξ gives rise to an economy in which it is impossible to produce fast enough so as to accomodate any situation where demand *ever* exceeds supply.

We shall return to closed economies in Subsections 2.6-2.7. Open economies are precisely those in which it is possible to respond to *every* netput $x \in D_0^J$ with an efficient (least) production plan which, furthermore, uniquely solves $DCP_Q(x)$. Formally,

Theorem 2.3. *If $\sigma(Q) < 1$ then Ψ_Q is well defined and Lipshitz continuous on D_0^J. Moreover, if $x \in D_0^J$ is upper-semi-continuous then $\Psi_Q(x)$ is a continuous process.*

Remarks:

2.5.A Theorem 2.3 is due to Harrison and Reiman [1981], who worked within C_0^J. They establish Lipshitz continuity of Ψ_Q on page 305. (They also show that when $\sigma(Q) \geq 1$, there must exist an $x \in D_0^J$ for which $DCP_Q(x)$ has no solution.) Their approach is based on the observation that $\underline{y} = \Psi_Q(x)$ is the unique fixed point of the mapping

$$y \to \sup_{0 \leq s \leq t} [Qy(s) - x(s)]^+, \quad t \geq 0,$$

which is a contraction to a high enough power. All the arguments of Harrison and Reiman extend to D_0^J, viewed as a Banach space with the locally uniform norm (corresponding to *u.o.c.* convergence). Hence, proofs of results for open systems will be omitted here (except for the second statement in Theorem 2.3, proved in Subsection 4.4).

2.5.B As already mentioned, netputs x in the applications that motivated our study are sample paths of a stochastic process. To guarantee that the regulator of a stochastic process is also a stochastic process, some form of measurability for Ψ_Q must hold. For open systems, Lipshitz continuity guarantees the measurability of $\Psi_Q : D_0^J \to D_0^J$, where both domain and range are endowed with the Kolmogorov σ-field generated by the cylinders (equivalently, the Borel σ-field associated with the Skorohod J_1-metric; in the sequel, all measurability statements concerning D_0^J, explicit or implicit, are with respect to this Kolmogorov σ-field.) Harrison and Reiman [1981] prove, in fact, that $\Psi_Q(x)(t)$ is a Lipshitz continuous, hence measurable, function of $\{x(s), 0 \leq s \leq t\}$, for all $t \geq 0$. (In standard terminology, $\Psi_Q(\cdot)$ is *adapted* to its argument, thus adding temporal measurability to the non-anticipativity 2.1.E.)

2.5.C The regulator mapping is often called the *oblique reflection* mapping. The reason for "oblique reflection" is the following (somewhat loose) geometric interpretation, where it is assumed, for simplicity, that Q is strictly column-substochastic ($e'Q < e'$) : the process $z = x + [I - Q]y$ "behaves" like x within the interior of the non-negative orthant $\mathbb{R}_+^J$; when z hits the boundary face $\{z_j = 0\}$, thus "threatening" to go negative, it is reflected instantaneously towards the interior of $\mathbb{R}_+^J$; the 'direction of reflection is the j-th column of the matrix $I - Q$ (which, indeed, points to the interior because $diag(I - Q) > 0$); finally, reflection is accomplished by the least increase in y_j which maintains the nonnegativity of z intact. (The Introduction to Harrison [1985] contains further comments on terminology.)

2.6 Recall that a Leontief economy is *closed* if $\sigma(Q) = 1$ and Q is irreducible. Irreducibility amounts to every commodity being a prerequisite for producing all the other commodities. To understand the implications of $\sigma(Q) = 1$, consider first a consumption matrix Q which is column-stochastic ($e'Q = e'$) with $diag(Q) = 0$. Then, endogenous production of a single unit of commodity j requires a *total* of exactly one unit from all other commodities $i \neq j$. For a general Q with $\sigma(Q) = 1$, let $\xi > 0$ be a left Peron-Fröbenius eigenvector of Q, namely $\xi' Q = \xi'$. Define $\Lambda = diag(\xi)$, and observe that $\tilde{Q} = \Lambda Q \Lambda^{-1}$ is column-stochastic and irreducible. Thus, after changing units in which commodities are measured, via $\Lambda = diag(\xi)$ as in the previous subsection, we may and *shall* assume that the consumption matrix of a closed system is, in fact, column-stochastic.

When $e'Q = e'$, a multiplication of 2.4.A from the left by e' reveals that any $x \in D_0^J$ with non-empty $\pi_Q(x)$ has the property $e'x(t) \geq 0$, for all $t \geq 0$. The example subsequent to Proposition 2.2 exhibits netputs x such that $e'x \equiv 0$ and $DCP_Q(x)$ has many solutions. In Subsection 4.3 we prove

Proposition 2.4. *Let Q be column-stochastic, irreducible, and let $x \in D_0^J$ be such that $e'x(t) > 0$ for all $t \geq 0$. If $\pi_Q(x) \neq \emptyset$, then the least element $\underline{y}$ of $\pi_Q(x)$ is the unique solution to $DCP_Q(x)$, thus $\underline{y} = \Psi_Q(x)$.*

We cannot prove as sharp a statement as Theorem 2.3 for closed networks. (See, however, Proposition 3.1.) Towards formulating a partial analog, define the subset $D_>^J$ of D_0^J by

$$D_>^J = \bigcap_{t>0} \bigcup_{\epsilon>0} D_{+\epsilon}^J\,[0,t]\,, \tag{2.4}$$

where $D_{+\epsilon}^J[0,t]$ denotes the set of all $x \in D_0^J$ that satisfy $e'x(s) \geq \epsilon$ for $s \in [0,t]$. We shall distinguish in $D_>^J$ the subset $D_>^{J,f}$ of functions that perform a *finite* number of jumps in finite time-intervals. As usual, $C_>^J$ stands for the continuous functions in $D_>^J$, which turns out to be

$$C_>^J = \{x \in C_0^J : e'x(t) > 0 \ \text{ for all } \ t \geq 0\}.$$

In Subsection 4.8 it is proved that

Theorem 2.5. *If Q is column-stochastic and irreducible, then Ψ_Q is well defined on $D_>^{J,f}$ and continuous on $C_>^J$. Moreover, $\Psi_Q(x)$ is a continuous process when $x \in D_>^J$ is upper-semi-continuous.*

Remark: Consider a sequence $\{x_n\}$, contained in the domain of Ψ_Q and converging *u.o.c.* to $x \in C_>^J$. By our definition of continuity at x, the sequence $\{\Psi_Q(x_n)\}$ converges *u.o.c.* to $\Psi_Q(x)$ *regardless* of whether $\{x_n\}$ satisfies any of the extra conditions in Proposition 2.4 or Theorem 2.5.

2.7 Theorem 2.5 is unsatisfactory in that it is easy to construct netput processes x which belong to the domain of Ψ_Q, yet they do not necessarily have a finite number of jumps on finite intervals. For such an example, consider $x \in D_=^J$ of the form

$$x(t) = x(0) + [Q - I]m(t)\,, \quad t \geq 0\,, \tag{2.5}$$

where Q is column-stochastic irreducible, $x(0) \in \mathbb{R}_+^J$ satisfying $e'x(0) > 0$, and $m \in D_\uparrow^J$ (*m can* jump an infinite number of times in finite intervals.) It is immediate that $y = m \in \pi_Q(x)$, hence $\pi_Q(x)$ has a least element $\underline{y}$. Since $e'x(t) > 0$, $t \geq 0$, Proposition 2.4 guarantees that in fact $\underline{y} = \Psi_Q(x)$.

Netputs of the form (2.5) are treated in Section 3. There, m is a sample path of either a multi-dimensional subordinator with finite first moments (Levy process with non-decreasing sample paths), or a multi-dimensional Brownian motion (in which case $m \in C_0^J$ replaces $m \in D_\uparrow^J$.) A subordinator m, unless deterministic-linear or compound-Poisson in nature, must perform an infinite number of jumps in finite time-intervals. The resulting x in (2.5) is, of course, locally of bounded variation. In contrast, a Brownian motion m has continuous sample paths of unbounded variation in every finite interval. The resulting x is a degenerate Brownian motion on a hyperplane.

We now formulate a theorem that, when combined with Theorem 2.7 in Subsection 2.9, accomodates all the netputs that arise here in Section 3, and in CMa-CMb. The theorem is formulated in terms of stochastic processes, thus avoiding explicit mention of some measurability issues that must be addressed in its proof.

Theorem 2.6. *Let Q be column-stochastic irreducible, and consider a stochastic process $X = \{X(t),\ t \geq 0\}$ whose sample paths satisfy*

$$e'X(t) > 0, \quad \textit{for all } t \ \geq 0.$$

If, furthermore, the sample paths of X are continuous (respectively, locally of bounded variation) with probability one, then $\Psi_Q(X)$ is a well-defined stochastic process that is adapted to X; it is continuous (respectively, locally of bounded variation) together with X.

Uniqueness of a solution to $DCP_Q(x)$, with x being a sample path of X above, is guaranteed by Proposition 2.4. Existence for such $x's$, namely $\pi_Q(x) \neq \emptyset$, is established for continuous x's in Theorem 2.5, and for $x's$ locally of bounded variation in Proposition 3.1. Adaptedness is a consequence of the pasting procedure employed in the proof of Theorem 2.5. This procedure amounts to a nonanticipative partitioning of $[0,\infty)$ to time-intervals in a way that Theorem 2.3 is applicable to each of them separately. The adaptedness, which holds on each of these time-intervals (recall Remark 2.5.B) can be "pasted" to hold on all of $[0,\infty)$. (Actually, any pasting procedure, for example the one employed by Harrison, Williams and Chen [1989, Section 5], will work as well; this is because, for the sample paths x under consideration, existence and uniqueness for $DCP_Q(x)$ are guaranteed *a priori.*)

2.8 The regulator mapping Ψ_Q enjoys interesting mathematical properties. We now give an account of those which have turned out to be most useful. It is convenient to introduce separate notation for the inventory process z associated with the least production plan $y = \Psi_Q(x)$. To this end, define the mapping Φ_Q, from the domain of Ψ_Q to D_+^J, by

$$\Phi_Q(x) = x + [I-Q]\Psi_Q(x) . \tag{2.6}$$

We now have

2.8.A (Monotonicity) If x^1, x^2 are both in the domain of Ψ_Q and $x^1 \leq x^2$, then $\Psi_Q(x^1) \geq \Psi_Q(x^2)$.

To prove 2.8.A observe that $\pi_Q(x^2)$ contains $\pi_Q(x^1)$, then use the characterization of $\Psi_Q(x^1)$ and $\Psi_Q(x^2)$ as least elements.

2.8.B (Adaptedness) $\Psi_Q(x)$ and $\Phi_Q(x)$ are non-anticipating with respect to the x's in their common domain.

Non-anticipativity was established in Proposition 2.1. By "adaptedness" (see 2.5.B) we mean non-anticipativity supplemented with the appropriate measurability. While we have not proved 2.8.B with "adapted" instead of "non-anticipating", Theorems 2.3, 2.6 and 2.7 suffice for our needs.

2.8.C (Rescaling by scalars) For any positive scalars α, β, and $t \geq 0$ we have

$$\Psi_Q[\alpha x(\beta\,\cdot)](t) = \alpha \Psi_Q(x)(\beta t) , \quad \text{and}$$
$$\Phi_Q[\alpha x(\beta\,\cdot)](t) = \alpha \Phi_Q(x)(\beta t) .$$

The argument on the left is the function $\{\alpha x(\beta t),\ t \geq 0\}$. For a proof, just verify 2.4.A-2.4.C and use uniqueness.

2.8.D (Shift) For all $s, t \geq 0$,

$$y(t+s) = y(s) + \Psi_Q[z(s) + x(s+\cdot) - x(s)](t) , \quad \text{and}$$
$$z(t+s) = \Phi_Q[z(s) + x(s+\cdot) - x(s)](t) ,$$

where $y(t) = \Psi_Q(x)(t)$, $z(t) = \Phi_Q(x)(t)$, and the argument on the right hand side above is the function $\{z(s) + x(s+t) - x(s),\ t \geq 0\}$. The proof of 2.8.D, as well as that of 2.8.E below is, again, simple verification.

2.8.E (Concatenation) Let $x \in D_0^J$, $\tau_1 > 0$ and $\tau_2 > 0$. Suppose that y^1 satisfies 2.4.A-2.4.C with data x on the time interval $[0, \tau_1]$, and that y^2 satisfies 2.4.A-2.4.C with data $x(\tau_1 + \cdot) + [I-Q]y^1(\tau_1)$ on $[0, \tau_2]$. Then

$$y(t) = \begin{cases} y^1(t) , & t \in [0, \tau_1) , \\ y^1(\tau_1) + y^2(t - \tau_1) , & t \in [\tau_1, \tau_1 + \tau_2] , \end{cases}$$

satisfies 2.4.A-2.4.C with data x on $[0, \tau_1 + \tau_2]$.

2.8.F Consider an open system ($Q \geq 0$, $\sigma(Q) < 1$), and let $y = \Psi_Q(x)$. Then there exists a positive constant c, depending only on Q, such that for all $0 \leq s \leq t$,

$$0 \leq y(t) - y(s) \leq c \cdot \sup_{s \leq u \leq v \leq t} [x(u) - x(v)]^+ . \tag{2.7}$$

We discuss (2.7) at the end of Subsection 4.5.

2.8.G Let $z = \Phi_Q(x)$, for either an open or a closed system. . Then for all $0 \leq s \leq t$,

$$z(t) - z(s) \leq \sup_{s \leq u \leq t} [x(t) - x(u)]^+ , \tag{2.8}$$

which implies

$$\sup_{s \leq u \leq v \leq t} [z(v) - z(u)] \leq \sup_{s \leq u \leq v \leq t} [x(v) - x(u)]^+ . \tag{2.9}$$

The inequalities (2.8)-(2.9) are established in Subsection 4.5.

Remark: The inequalities (2.7)-(2.9) remain intact if, on their right-hand-sides, the positive-part function is either omitted or replaced by the absolute-value function.

2.9 So far, our Leontief economies have evolved in an infinite-horizon time-set, namely within $D^J[0, \infty)$. However, all that has been said so far, in particular 2.8.A-2.8.G, still prevails up to a finite horizon, namely within $D^J[0, \tau)$ for any $\tau > 0$. The modifications are minor. As an example, the definition (2.1) of a production set $\pi_Q(x), x \in D_0^J[0, \tau)$, then becomes

$$\pi_Q(x) = \{y \in D_\uparrow^J[0, \tau) : x(t) + [I-Q]y(t) \geq 0,\ 0 \leq t < \tau\} ,$$

solving $DCP_Q(x)$ for $x \in D^J[0,\tau)$ amounts to finding a regulator in $D^J_\uparrow[0,\tau)$ that satisfies 2.4.A-2.4.C on $[0,\tau)$, and so on. Restricting attention to a finite horizon turns out a necessity for netputs x at which a regulator $y \in D^J_\uparrow[0,\tau)$ can not be guaranteed to either exist or be unique beyond $\tau < \infty$. We shall then explicitly indicate that $\Psi_Q(x) = y$ on $[0,\tau)$.

The following finite-horizon scenario arises in the proof of both Theorem 3.4 here and Theorem 4.1 in CMb.

Theorem 2.7. *Let Q be column-stochastic irreducible. Suppose that, on some probability space, there are three sequences of stochastic processes $X^n \in B^J_0$, $Y^n \in D^J_\uparrow$, and $Z^n \in D^J_+$, which conform to the following:*

2.9.A $\quad e'X^n \geq 0$ *for all* $n = 1,2,\dots$;

2.9.B *The sequence of stopping times*

$$\tau_n = min\{t \geq 0 : e'X^n(t) = 0\}$$

converges a.s. to infinity, as $n \to \infty$;

2.9.C *On $[0,\tau_n)$, $Y^n = \Psi_Q(X^n)$ and $Z^n = \Phi_Q(X^n)$, for all $n = 1,2,\dots$.*

If in addition,

$$X^n \to X \quad u.o.c.\ a.s.\,,$$

where X is a stochastic process with sample paths in $C^J_>$, then

$$Y^n \to \Psi_Q(X) \quad and \quad Z^n \to \Phi_Q(X), \quad u.o.c.\ a.s.\,,$$

as $n \to \infty$.

The ingredients that constitute the proof of Theorem 2.7 have all been established, except for

Lemma 2.8. *Let Q be column-stochastic irreducible. Consider a sequence $\{x_n\}$ that converges u.o.c. to $x \in C^J_>$. Suppose that $y_n = \Phi_Q(x_n)$ on $[0,\tau_n)$, where $\tau_n \to \infty$ as $n \to \infty$. Then, $\{y_n\}$ converges u.o.c. to $y = \Psi_Q(x)$.*

Proof: Fix $T > 0$. There exists an N such that $\tau_n > T$ for all $n \geq N$. From $y_n = \Psi_Q(x_n)$ on $[0,T]$ for $n \geq N$, and the continuity of Ψ_Q on $C^J_>[0,T]$, follows that $y_n \to y = \Psi_Q(x)$ uniformly on $[0,T]$. The observation that T is arbitrary now completes the proof.

3. RBV's and RBM's

The present section is devoted mainly to economies driven by netputs x that are locally of bounded variation ($x \in B^J_0$). The focus is on closed economies. In particular, we introduce a stochastic Leontief economy whose total inventory remains constant in time; its netput is a Levy process, with sample paths in $B^J_=$ (Example 3.8). As this constant total inventory increases indefinitely (the economy becomes heavily loaded), it is demonstrated that the stochastic economy, properly aggregated, converges strongly to a deterministic economy with linear netput (Theorem 3.3). This latter economy provides a deterministic first-order approximation to the former.

Through capturing long-run average behavior, the first-order approximation induces a partition of sectors/commodities into two sets (3.9.F). One set consists, roughly speaking, of commodities for which the demand regularly exceeds supply; typically, the inventory of such commodities is low, and the sectors that produce them are constantly busy. We call such sectors *unbalanced.* The other set includes those commodities whose exogenous supply, generally speaking, suffices to handle the demand for them; the corresponding sectors are rarely busy, and they function mainly as storage buffers. We call these latter sectors *balanced.*

Next, the deviations of the Leontief economy from its first-order approximation are quantified (Theorem 3.4). This is accomplished via weak convergence of the deviations. Their limit is described in terms of a closed RBM, which provides a stochastic second-order diffusion approximation to the original Leontief economy.

3.1 $\quad$ Suppose that a netput $x \in B^J_0$ has the form

$$x(t) = x(0) + a(t) + [Q - I]m(t)\ \ ,\ t \geq 0\,, \tag{3.1}$$

where $x(0) \geq 0$, and $a, m \in D^J_\uparrow$. Then the process $y = m$ is a feasible production plan responding to such an x, because $z = x + [I - Q]m = x(0) + a \in D^J_+$, and consequently $\pi_Q(x) \neq \emptyset$. In an open system ($\sigma(Q) < 1$), every netput $x \in B^J_0$ is of the form (3.1), (which reconfirms the existence part of Theorem 2.3). Indeed, $x \in B^J_0$ can be represented as

$$x(t) = x(0) + x^1(t) - x^2(t)\ ,\ \ t \geq 0\,, \tag{3.2}$$

with $x^1, x^2 \in D^J_\uparrow$. Since $I - Q$ is non-singular with a non-negative inverse, taking $a = x^1$, $m = [I - Q]^{-1}x^2$ yields (3.1).

3.2 From now on, we shall consider only closed models (Q column-stochastic, irreducible). As already explained prior to Proposition 2.4, for $\pi_Q(x) \neq \emptyset$ it is necessary that $e'x \geq 0$. But in fact,

Proposition 3.1. *Let $Q \geq 0$ be column-stochastic irreducible. Then for $x \in B_0^J$, the condition $e'x \geq 0$ is necessary and sufficient for $\pi_Q(x) \neq \phi$.*

Proposition 3.1 is an immediate consequence of

Lemma 3.2. *Consider $x \in B_0^J$ for which $e'x \geq 0$. Then x is of the form (3.1) with $x(0) + a \geq 0$, $a \in B_0^J$, $m \in D_\uparrow^J$. In particular, $e'x \equiv e'x(0)$ implies (3.1) with $a \equiv 0$.*

To deduce Proposition 3.1 note that, as in Subsection 3.1, $y = m$ in Lemma 3.2 belongs to $\pi_Q(x)$. Also note that $x \in B_0^J$ can not be relaxed to $x \in D_0^J$ in view of

3.2.A When $e'x \equiv 0$, $\pi_Q(x) \neq \phi$ if and only if $x \in B_0^J$.

Indeed, $z = x + [I - Q]y \geq 0$, $y \in D_\uparrow^J$, implies that $e'z = e'x + 0 = 0$, hence $z = 0$ and $x = [Q-I]y \in B_0^J$. (A plausible relaxation of $x \in B_0^J$ is to $x \in D_>^J$, as defined in (2.4), but we can not substantiate this yet.)

3.3 *Proof of Lemma 3.2:* Our starting point is the fact that the orthogonal complement of the row-span of $[Q - I]$ coincides with the one-dimensional span of the vector e. This implies that

3.3.A Every $x \in B_0^J$ with $e'x \equiv 0$ can be represented as

$$x = [Q - I]m^0 \ , \ \ m^0(0) = 0 \ , \ \ m^0 \in B^J \ . \tag{3.3}$$

It is now possible to modify m^0 in (3.3) to be in $D_\uparrow^J$. Simply take a positive Peron-Fröbenius right eigenvector v of Q; write m^0 as $m^1 - m^2$ with $m^1, m^2 \in D_\uparrow^J$; define

$$\delta(t) = \sum_{j=1}^{J} m_j^2(t)/v_j \in D_\uparrow^1 \ ,$$

and check that $\delta v - m^2 \in D_\uparrow^J$. Hence $x = [Q - I](m^1 - m^2 + \delta v)$, which shows that

3.3.B Every $x \in B_0^J$ with $e'x \equiv 0$ can be represented as $x = [Q - I]m$, $m \in D_\uparrow^J$.

We now reduce the case $e'x \geq 0$ to 3.3.B. Suppose first that $e'x(0) > 0$, and define

$$\tilde{x}(t) = x(t) - \frac{e'x(t)}{e'x(0)} x(0) \ , \ \ t \geq 0 \, .$$

Since $\tilde{x}(0) = 0$ and $e'\tilde{x} \equiv 0$, 3.3.B guarantees the existence of an $m \in D_\uparrow^J$ such that $\tilde{x} = [Q - I]m$. Then, x is of the form (3.1) with $a(t) = [e'x(t)/e'x(0) - 1]x(0)$, $t \geq 0$. In case $e'x(0) = 0$, we apply 3.3.B to $\tilde{x} = x - (e'x)e/J$, which yields $a = (e'x)e/J$. Lemma 3.2 has now been proved.

3.3.C *Remark:* Suppose that x in Lemma 3.2 is a sample path of a stochastic process. The construction of a and m is explicit enough to demonstrate that both can be taken to be adapted to x (which entails that $a(t)$ and $m(t)$ be measurable functions of $\{x(s), s \leq t\}$, for all $t \geq 0$).

3.4 Let $X = \{X(t), t \geq 0\}$ be a stochastic process with sample paths in $B_=^J$. Applying Lemma 3.2 to each sample path of X yields the representation

$$X(t) = X(0) + [Q - I]M(t) \ , \ \ t \geq 0 \, . \tag{3.4}$$

Here $M = \{M(t), t \geq 0\}$ is a stochastic process whose sample paths are in $D_\uparrow^J$, and which is adapted to X in view of Remark 3.3.C. Rewriting (3.4) as $X(0) = X(t) + [I - Q]M(t)$, $t \geq 0$, exhibits M as a (stochastic) feasible production plan; it responds adaptively to X by maintaining a constant inventory profile $X(0)$ at all times. Typically, this does not constitute an efficient operation, but the existence of one is guaranteed by Proposition 2.1, and its uniqueness by Proposition 2.4. With the help of the discussion subsequent to Theorem 2.6, one now deduces the existence of a unique pair of stochastic processes Y, Z, both adapted to the given $X \in B_=^J$, such that

3.4.A $Z = X + [I - Q]Y$,

3.4.B $Z \in D_+^J$, $Y \in D_\uparrow^J$,

3.4.C Y is the least production plan responding to X,

or equivalently to 3.4.C,

3.4.D Y is complementary to Z: $\int_0^\infty Z_j \, dY_j = 0$, for $j = 1, \ldots, J$.

The relations 3.4.A–3.4.D, which are to be interpreted pathwise, can be summarized as

$$Y = \Psi_Q(X) \quad \text{and} \quad Z = \Phi_Q(X).$$

The sample paths of Z are in B^J_+; Z is constrained to evolve on the $\mathbb{R}^J$-simplex $\{z : z \geq 0, e'z = e'X(0)\}$, by reflecting/regulating it from the facets of the simplex towards its interior. As explained in 2.5.C, and as alluded to in our Introduction, we shall call Z a closed Reflected/Regulated process of Bounded Variation, or simply a *closed RBV*.

3.5 In view of Theorem 2.6, X with sample paths in $C^J_{=}$ still gives rise to unique Y, Z, that satisfy 3.4.A–3.4.D (even if the sample paths of X are locally of unbounded variation). In such circumstances, the representation (3.4) still prevails, but M is guaranteed only to be continuous with $M(0) = 0$. The stochastic process $Z = \Phi_Q(X)$, where X is of the form (3.4) with M being a J-dimensional Brownian motion, will be referred to as a *closed RBM*, (which stands for a closed Reflected/Regulated Brownian motion). The process X is a degenerate J-dimensional Brownian motion with sample paths on the hyperplane $\{x \in \mathbb{R}^J : e'x = e'X(0)\}$.

3.6 A Brownian motion X in a hyperplane *cannot* directly represent physical netput (recall Remark 2.1.C; see also Chapter 2 in Harrison [1985] for further commentary). Due to the continuity of the regulator mapping, however, RBM's do have a physical interpretation: they provide diffusion approximations to RBV's whose netputs, properly centered and rescaled, can be approximated by Brownian motions.

The role of multidimensional RBM's as diffusion approximations first emerged in Harrison [1978], who approximated tandem queues in heavy traffic. A generalization to balanced nonparametric open Jackson networks was carried out by Reiman [1984], and then extended in Johnson [1983] to partially cover open *un*balanced models. CMa and CMb completed Johnson's extension, and supplemented it with diffusion approximations for heavily loaded closed networks. We now review some of these approximations, in the context of heavily-loaded stochastic closed Leontief economies.

3.7 Consider a sequence of stochastic closed Leontief economics, indexed by $n = 1, 2, \ldots$. The n-th economy is driven by a stochastic netput $X^n \in B^J_{=}$ with the following two properties:

3.7.A $\quad e'X^n(t) \equiv n \quad , \quad t \geq 0\,,$

3.7.B $\quad \bar{X}^n(t) = \frac{1}{n}X^n(nt) \to \bar{X}(t)$ *u.o.c* almost surely (*a.s.*).

Here $\bar{X} \in C^J_1$ is assumed to have the form

$$\bar{X}(t) = \bar{X}(0) + \xi t \quad , \quad t \geq 0\,, \tag{3.5}$$

where $\bar{X}(0)$ is (necessarily) an element of the unit simplex, and ξ belongs to the hyperplane $H = \{x \in \mathbb{R}^J : e'x = 0\}$. (The convergence 3.7.B amounts to a Functional Strong Law of Large Numbers (FSLLN) which the sample paths of X^n are assumed to adhere to; this law captures the long-run average trend in the evolution of X^n, as will be demonstrated in (3.9) below). We further assume, but only for simplicity, that all economies operate according to the same (irreducible column-stochastic) consumption matrix Q. Denoting the efficient plan responding to X^n by $Y^n = \Psi_Q(X^n)$, the corresponding inventory process is then $Z^n = \Phi_Q(X^n) = X^n + [I - Q]Y^n$. The ***total*** inventory that circulates in the n-th economy equals n units; formally, $e'Z^n(t) \equiv n$, for all $t \geq 0$, $n = 1, 2, \ldots$. We are interested in approximating the n-th economy for large n (heavy loading).

3.8 *An Example:* The conditions 3.7.A–3.7.B are met by X^n's of the form

$$X^n(t) = n\bar{X}(0) + [Q - I]M(t) \quad , \quad t \geq 0\,, \tag{3.6}$$

where $\bar{X}(0)$ and M are independent, $\bar{X}(0)$ takes values in the unit simplex, and M is an integrable Levy process with sample paths in $D^J_\uparrow$. Indeed, such a Levy process has a mean vector $\mu = EM(1)$, which is also recoverable from the SLLN (Gihman and Skorohod [1975], Theorem 9 on page 338)

$$\frac{M(t)}{t} \to \mu \quad , \quad \text{as } t \uparrow \infty \quad , \quad a.s.\,. \tag{3.7}$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n}M(nt) = \mu t \quad , \quad \text{for all } t \geq 0\,, \ a.s.\,. \tag{3.8}$$

But pointwise convergence of monotone functions to a continuous function is equivalent to *u.o.c.* convergence (Resnick [1987], Chapter 1, or the remark subsequent to the proof of Theorem 6.1 in Whitt [1980]). One concludes that the convergence in (3.8) is *u.o.c* almost surely; hence X^n in (3.6) satisfies 3.7.B and (3.5), with $\xi = [Q - I]\mu$.

The convergence (3.7) is actually equivalent to that in (3.8), as shown in Corollary 2 of Kingman [1963]. Either implies that X^n in (3.6) satisfies

$$\frac{1}{t}X^n(t) \to \xi \quad \text{as } t \uparrow \infty\,, \ a.s.\,, \tag{3.9}$$

thus formalizing our previous claim that 3.7.B captures long-run average trend.

With netputs of the form (3.6), the procedure of increasing n to infinity amounts to subjecting a *fixed* economy to heavy loading. This is accomplished by increasing the ***total initial*** inventory, while maintaining its relative distribution among sectors intact. ($\bar{X}_j(0)$ is the fraction of total initial inventory that is present in sector $j = 1, \ldots, J$).

3.9 Define the rescaled processes

$$\bar{Y}^n(t) = \frac{1}{n}Y^n(nt) \ , \quad \bar{Z}^n(t) = \frac{1}{n}Z^n(nt) \ , \quad t \geq 0 ,$$

for $n = 1, 2, \ldots$. By property 2.8.C of the regulator mapping,

$$\bar{Y}^n = \Psi_Q(\bar{X}^n) \ , \quad \bar{Z}^n = \Phi_Q(\bar{X}^n) \ , \quad n = 1, 2, \ldots$$

The continuity of the regulator (Theorem 2.5) implies the FSLLN

3.9.A $\quad \bar{Y}^n \to \bar{Y} = \Psi_Q(\bar{X})$ *u.o.c a.s.*, and

3.9.B $\quad \bar{Z}^n \to \bar{Z} = \bar{X} + [I - Q]\bar{Y}$ *u.o.c a.s.*,

where $\bar{X}$ is given in (3.5). Let us refer to the deterministic economy driven by the time-linear netput $\bar{X}$ as "the linear economy". Every stochastic economy driven by X^n, n large enough, will be called "the heavily loaded stochastic economy". (This purposely vague usage of wording does facilitate terminology; it is close to being accurate when X^n has the form (3.6).)

As already discussed, the linear economy provides a first order approximation to the heavily-loaded stochastic economy (via aggregating both its time and inventory units). The main role of the linear model is to expose balanced and unbalanced sectors. These two concepts will now be formally introduced, by adapting Theorem 5.2 in CMa to the Leontief setting.

The linear economy reaches equilibrium in the sense that there exists a time $\tau < \infty$, after which both inventory levels $\bar{Z}$ and efficient production ***rates***, say $\underline{\eta} = (\underline{\eta}_j)$, remain constant in time. (Until τ, levels and rates vary in a piecewise-linear fashion). The equilibrium production rate vector $\underline{\eta}$ is the ***least*** $\eta \geq 0$ that satisfies $\xi + [I-Q]\eta \geq 0$, where ξ is given in (3.5). Equivalently, $\underline{\eta}$ emerges as the *least* solution to the Linear Complementarity Problem with matrix $I - Q$ and data vector ξ. This problem will be denoted by $LCP_Q(\xi)$, and solving it constitutes finding a pair of J-dimensional vectors η, ζ, such that

3.9.C $\quad \zeta = \xi + [I - Q]\eta$,

3.9.D $\quad \zeta \geq 0, \eta \geq 0$,

3.9.E $\quad \zeta \cdot \eta = 0 \quad$ (equivalently $\eta_j = 0$ when $\zeta_j > 0$.)

With Q that is irreducible column-stochastic, $LCP_Q(\xi)$ has a solution if and only if $e'\xi \geq 0$; uniqueness of this solution is equivalent to $e'\xi > 0$. (Proofs are given in Subsection 4.1.) For ξ in (3.5), $e'\xi = 0$. In this case, $LCP_Q(\xi)$ has many solutions, but the $\underline{\eta}$ we seek is least, hence unique. (In contrast to η, ζ is detemined uniquely by 3.9.C.-3.9.E; indeed $\zeta = 0$, namely

$$\xi = [Q - I]\eta , \tag{3.10}$$

because $e'\zeta = e'\xi + [e' - e'Q]\eta = 0$ and $\zeta \geq 0$.)

Call sector j a ***balanced*** production sector if $\underline{\eta}_j = 0$, and ***unbalanced*** if $\underline{\eta}_j > 0$. (It turns out that there is at least one balanced sector.) Denote by α and $\beta \neq \emptyset$ the partition of $\{1, \ldots, J\}$ into sets of unbalanced and balanced sectors respectively. Then

Theorem 3.3. *There exists a time $\tau < \infty$, after which $\bar{Z}$ and $\bar{Y}$ in 3.9.A–3.9.B are given by*

$$\bar{Z}_\alpha(t) = 0 \quad , \quad t \geq \tau , \tag{3.11}$$

$$\bar{Z}_\beta(t) = \bar{X}_\beta(0) + Q_{\beta\alpha}[I - Q_\alpha]^{-1}\bar{X}_\alpha(0) \quad , \quad t \geq \tau , \tag{3.12}$$

$$\bar{Y}_\alpha(t) = \bar{Y}_\alpha(\tau) + \underline{\eta}_\alpha(t - \tau) \quad , \quad t \geq \tau , \text{ and} \tag{3.13}$$

$$\bar{Y}_\beta(t) = 0 \quad , \quad t \geq 0 . \tag{3.14}$$

In order that $\tau = 0$, it is necessary and sufficient that $\bar{X}_\alpha(0) = 0$, in which case the above simplifies to

$$\bar{Z}(t) = \bar{X}(0) \quad , \quad \bar{Y}(t) = \eta t \quad , \quad t \geq 0 . \tag{3.15}$$

3.9.F *Remarks:* The existence of $[I - Q_\alpha]^{-1}$ is justified as in Lemma 4.4 of CMa. The relations (3.11)-(3.14) formalize our previous discussion about two types of sectors: the unbalanced sectors α, which constantly produce ($\underline{\eta}_\alpha > 0$ in 3.13), and whose inventory is empty (3.11); and the balanced sectors, which do not produce (3.14), and whose role is to serve as storage buffers (3.12).

3.10 Theorem 3.3 establishes, via the FSLLN 3.9.A–3.9.B, the deterministic linear economy (driven by $\bar{X}$ in (3.5)) as a first-order approximation to the stochastic economy (driven by X^n, n large). We now proceed to derive a second-order stochastic approximation, based on a Functional Central Limit Theorem (FCLT) analog of 3.7.B. The second-order approximation quantifies deviations of the Leontief economy from its first-order approximation.

Consider again the sequence of economies described in Subsection 3.7. Instead of the FSLLN 3.7.B, assume now that the netputs conform to the following FCLT in D^J:

3.10.A $\hat{X}^n(t) = \frac{1}{n}X^n(n^2t) - \xi nt \xrightarrow{d} \hat{X}(t)$,

with $\xi \in H$ as before, and the weak limit $\hat{X} = \{\hat{X}(t),\, t \geq 0\}$ being a stochastic process with *continuous* sample paths, each of which (necessarily) belongs to $\hat{X}(0) + H$ (H translated by $\hat{X}(0)$). The FCLT 3.10.A implies that

3.10.B $\bar{X}^n(t) = \frac{1}{n}X^n(nt) \xrightarrow{d} \hat{X}(0) + \xi t$ in D^J,

which is a Functional Weak Law of Large Numbers in D^J. To see that, let $\tau^n(t) = t/n$, $t \geq 0$, be a sequence of time-changes (Billingsley [1968], Section 17, or Whitt [1980], Section 6). Then, $\tau^n \to 0$ *u.o.c.*, combined with 3.10.A, implies that $\hat{X}^n[\tau^n(t)] \xrightarrow{d} \hat{X}(0)$, hence 3.10.B.

Back to Example 3.8: Add the assumption that M in (3.6) is, in fact, *square*-integrable, with covariance matrix Γ that equals *cov* $M(1)$. It follows (from Theorem 1.4 on page 339 of Ethier and Kurtz [1986], or Aldous [1989], for example) that

$$\frac{1}{n}M(n^2t) - \mu nt \xrightarrow{d} \hat{M}(t) \quad in \quad D^J ,$$

where $\mu = EM(1)$, and $\hat{M}$ is a J-dimensional driftless Brownian motion with covariance matrix Γ and $\hat{M}(0) = 0$. Consequently, X^n in (3.6) satisfies 3.10.A, with $\xi = [Q-I]\mu$ and $\hat{X} = \bar{X}(0)+[Q-I]\hat{M}$; $\hat{X}$ is a driftless Brownian motion with covariance matrix $[Q-I]'\Gamma[Q-I]$. (Note that, in this Brownian model, 3.10.B holds in its strong version, where convergence is *u.o.c* almost surely.)

3.11 The sequences for which a FCLT will be derived are given, for

$n = 1, 2, \ldots$, by

$$\hat{Z}^n(t) = \frac{1}{n}Z^n(n^2t) \quad , \quad t \geq 0 , \quad \text{and}$$
$$\hat{Y}^n(t) = \frac{1}{n}Y^n(n^2t) - \eta nt \quad , \quad t \geq 0 .$$

These two sequences are related, due to (3.10), through

$$\hat{Z}^n = \hat{X}^n + [I-Q]\hat{Y}^n . \tag{3.16}$$

We first partition (3.16) into blocks α and β to get

$$\hat{Z}^n_\alpha = \hat{X}^n_\alpha + [I-Q_\alpha]\hat{Y}^n_\alpha - Q_{\alpha\beta}\hat{Y}^n_\beta , \tag{3.17}$$

$$\hat{Z}^n_\beta = \hat{X}^n_\beta - Q_{\beta\alpha}\hat{Y}^n_\alpha + [I-Q_\beta]\hat{Y}^n_\beta . \tag{3.18}$$

In view of 3.9.F, one can isolate $\hat{Y}^n_\alpha$ in (3.17). Substituting the outcome into (3.18) results in

$$\hat{Z}^n_\beta = \tilde{X}^n + [I-\tilde{Q}]\hat{Y}^n_\beta , \tag{3.19}$$

where

$$\tilde{X}^n = \hat{X}^n_\beta + Q_{\beta\alpha}[I-Q_\alpha]^{-1}(\hat{X}^n_\alpha - \hat{Z}^n_\alpha) , \tag{3.20}$$

and

$$\tilde{Q} = Q_\beta + Q_{\beta\alpha}[I-Q_\alpha]^{-1}Q_{\alpha\beta} \tag{3.21}$$

is an irreducible column-stochastic matrix of order $|\beta|$. (Consult CMa, especially Lemma 4.3, for properties and interpretations of $\tilde{Q}$).

Theorem 3.4. *Consider the sequence of closed economies from Subsection 3.7. Assume that 3.7.A and 3.10.A hold. Then the weak convergence*

$$(\hat{Y}^n, \hat{Z}^n) \xrightarrow{d} (\hat{Y}, \hat{Z}) \quad in \quad t > 0 , \tag{3.22}$$

holds as $n \uparrow \infty$. The limit is described for $t > 0$ by

$$\hat{Z}_\beta = X + [I-\tilde{Q}]\hat{Y}_\beta , \tag{3.23}$$
$$X = \hat{X}_\beta + Q_{\beta\alpha}[I-Q]^{-1}\hat{X}_\alpha \in C_1^{|\beta|} , \tag{3.24}$$
$$\hat{Y}_\beta = \Psi_{\tilde{Q}}(X) , \tag{3.25}$$
$$\hat{Z}_\alpha = 0 , \tag{3.26}$$
$$\hat{Y}_\alpha = [I-Q_\alpha]^{-1}(Q_{\alpha\beta}\hat{Y}_\beta - \hat{X}_\alpha) , \tag{3.27}$$

where β and α are, respectively, the balanced ($\beta \neq \emptyset$, $\eta_\beta = 0$) and unbalanced ($\eta_\alpha > 0$) production sectors.

Concluding Example 3.8: Under square integrability of M, the process X is a $|\beta|$-dimensional driftless Brownian motion that evolves on the $(|\beta| - 1)$-dimensional hyperplane $\{x \in \mathbb{R}^{|\beta|}: e'x = 1\}$; the inventory process $\hat{Z}_\beta$ at the balanced production sectors is, then, a β-dimensional closed RBM on the unit simplex.

3.12 One can reduce the proof of Theorem 3.4 to that of Theorem 4.1 in CMa. We now review the main steps of the proof, while interweaving some commentary as we go along.

As a preliminary step, the Skorohod Representation theorem (Skorohod [1956]) allows one to assume that the netputs X^n are all supported on a single probability space, in which the convergence 3.10.A, hence also 3.10.B, hold uniformly on compact subsets of $[0,\infty)$ almost surely. This reduces the proof of (3.22) to *u.o.c.* convergence for individual sample paths. First one establishes (3.26), namely

$$\hat{Z}^n_\alpha \to 0 \quad \text{in} \quad t > 0 \quad , \quad u.o.c.\,. \tag{3.28}$$

Note that $\hat{Z}^n_\alpha(0) \to \hat{X}_\alpha(0)$, hence (3.28) cannot be expected to hold at $t = 0$ unless, of course, $\hat{X}_\alpha(0) \equiv 0$ (negligible initial inventory at the unbalanced sectors). In order to simplify the discussion, let us assume that indeed $\hat{X}_\alpha(0) \equiv 0$. Then (3.28), with $t \geq 0$, implies that $\tilde{X}^n$ in (3.20) converges *u.o.c.* to X in (3.24), and furthermore,

$$e'\tilde{X}^n = e'\hat{Z}^n_\beta = 1 - e'\hat{Z}^n_\alpha \to 1 \quad u.o.c.\ . \tag{3.29}$$

The convergence (3.29) guarantees that $X \in C_1^{|\beta|}$, and that $\tau^n = min\{t \geq 0: e'\tilde{X}^n_\beta = 0\}$ increases indefinitely with n. Recalling that $\eta_\beta = 0$, one can now interpret (3.19) as

$$\hat{Y}^n_\beta = \Psi_{\tilde{Q}}(\tilde{X}^n) \quad , \quad \hat{Z}^n_\beta = \phi_{\tilde{Q}}(\tilde{X}^n) \quad \text{on} \quad [0,\tau^n). \tag{3.30}$$

By Theorem 2.7, $\hat{Y}^n_\beta$ in (3.30) converges to $\hat{Y}_\beta$ in (3.25), hence $\hat{Z}^n_\beta$ converges to $\hat{Z}_\beta$ in (3.23). Finally, letting $n \uparrow \infty$ in (3.17) establishes the convergence of $\hat{Y}^n_\alpha$ to $\hat{Y}_\alpha$ in (3.27). The outline of the proof is now complete. (Readers interested in details or further commentary should consult Subsections 4.1–4.3 in CMb.)

4. Proofs that Support Section 2

4.1 In this first subsection, the relation between DCP and LCP will be elaborated. We start by tailoring to our needs some known results from finite-dimensional mathematical programming.

For a vector $a \in \mathbb{R}^J$, and an irreducible column-stochastic $J \times J$ matrix Q, let $\pi_Q(a) = \{b \geq 0 : a + [I - Q]b \geq 0\}$. (No confusion should arise with $\pi_Q(x)$ in (2.1).) The linear complementarity problem, $LCP_Q(a)$, is to identify $b \in \pi_Q(a)$ that satisfies the complementarity condition $b'\{a + [I - Q]b\} = 0$; (Cottle and Dantzig [1968]; our present formulation is clearly equivalent to 3.9.C-3.9.E). The least-element problem is to find in $\pi_Q(a)$ a least element (Cottle and Veinott [1972]).

Proposition 4.1. *For $a \in \mathbb{R}^J$, the set $\pi_Q(a)$ is non-empty if and only if $e'a \geq 0$.*

Proof: If $\pi_Q(a)$ is non-empty, then $a + [I - Q]b \geq 0$ for some $b \geq 0$; hence, $e'a = e'\{a + [I - Q]b\} \geq 0$ since Q is column-stochastic, which proves "only if". The "if" part follows from Theorem 6 of Cottle and Stone [1983]. Alternatively, it suffices to demonstrate that the system of inequalities

$$w'[I - Q] \leq 0, \tag{4.1}$$
$$w'a < 0, \tag{4.2}$$
$$w \geq 0, \tag{4.3}$$

is infeasible in w (Theorem 2.3 in Gale [1960], for example). Let $\xi > 0$ be a right-eigenvector of Q. Then $w'[I - Q]\xi = 0$ which, together with (4.1), implies that $w'[I - Q] = 0$. Being a left eigenvector of Q, it follows that $w = \theta e$ for some constant θ. But this is inconsistent with $e'a \geq 0$, in view of (4.2)-(4.3), and we are done.

Lemma 4.2. *Let $a \in \mathbb{R}^J$ be such that $e'a \geq 0$. Then $\pi_Q(a)$ has a least element which, furthermore, solves $LCP_Q(a)$.*

Proof: The set $\pi_Q(a)$ is non-empty by Proposition 4.1. It follows, as in Proposition 2.1, that $\pi_Q(a)$ has a least element b. (Alternatively, apply Cottle and Veinott [1972] to the Leontief matrix $[I - Q\,, I]$.) To show complementarity, put $c = a + [I - Q]b$. If $c'b \neq 0$, then there must exist an index i such that both $b_i > 0$, $c_i > 0$. Define $\tilde{b}_i = b_i - \epsilon$ and $\tilde{b}_j = b_j$ for $j \neq i$, with $\epsilon > 0$ small enough so that $\tilde{b} \geq 0$ and $\tilde{c} = a + [I - P]\tilde{b} \geq 0$. Since $\tilde{b} \leq b$ and $\tilde{b} \neq b$, b can not be least in $\pi_Q(a)$.

Proposition 4.3. *Let $a \in \mathbb{R}^J$. Then $LCP_Q(a)$ has a unique solution if and only if $e'a > 0$.*

Proof: One possible proof is to deduce the result from Cottle and Stone [1983], page 212. Alternatively, non-uniqueness when $e'a = 0$ follows along the lines of 3.3.A-3.3.B. Suppose now that $e'a > 0$. Lemma 4.2 guarantees that $\pi_Q(a)$ has a least element b which solves $LCP_Q(a)$. Let $\tilde{b} \geq b$ be another solution to $LCP_Q(a)$, and put $c = a + [I-Q]b$, $\tilde{c} = a + [I-Q]\tilde{b}$. Since $\tilde{c}'\tilde{b} = 0$, $\tilde{b}_i > b_i$ would imply that $\tilde{c}_i = 0 \leq c_i$. If $\tilde{b}_i = b_i$, then $\tilde{c}_i \leq c_i$ from their definition, and the fact that $\tilde{b} \geq b$. Therefore,

$$[I-Q](\tilde{b} - b) = \tilde{c} - c \leq 0.$$

By an argument similar to the proof of Proposition 4.1, $\tilde{b} - b = \theta\xi$ where θ is a non-negative constant, and $\xi > 0$ is a right-eigenvector of Q. But $0 \leq \tilde{c} \leq c$ and $c'b = 0$, hence

$$0 = \tilde{c}'\tilde{b} = \tilde{c}'b + \theta\tilde{c}'\xi = \theta\tilde{c}'\xi.$$

From $\tilde{b} \geq b$ and $\tilde{b} \neq b$ we get $\theta > 0$ which , together with $\theta\tilde{c}'\xi = 0$ and $\xi > 0$, implies that $\tilde{c} = 0$. Finally, $0 = \tilde{c} = a + [I-Q]\tilde{b}$ leads to $e'a = 0$, contradicting the initial assumption $e'a > 0$.

It will now be shown that, for any step function $x \in D^J_>$, solving $DCP_Q(x)$ amounts to iteratively solving LCP at the jump points of x. A step function $x \in D^J_>$ has the form

$$x(t) = \sum_{k=0}^{\infty} x(t_k) \cdot 1_{[t_k, t_{k+1})}(t),$$

where $t_0 = 0$, $t_k < t_{k+1}$ (for t_k finite), either $t_K = \infty$ for some K (in which case $t_k = \infty$ for all $k \geq K$) or $t_k \to \infty$ as $k \to \infty$; finally $x(0) \geq 0$ and $e'x(t_k) > 0$ for all $k = 0, 1, \ldots$. Consider a solution y to $DCP_Q(x)$, and put $z = x + [I-Q]y$. Clearly, for $0 = t_0 \leq t < t_1$ it must be that $y(t) = y(t_0) = 0$, and $z(t) = x(t_0) + [I-Q]y(t_0)$. In other words, on $[t_0, t_1)$ the pair (y, z) constantly equals the solution to $LCP_Q(x(t_0))$. Proceeding to t_1, and more generally to t_k, $k = 1, 2, \ldots$, the characterizing relations 2.4.A-2.4.C, combined with the shift property 2.8.D, imply that for $t \in [t_k, t_{k+1})$

$$z(t) = [z(t_k-) + x(t_k) - x(t_k-)] + [I-Q][y(t) - y(t_k-)], \tag{4.4}$$

$$y(t) - y(t_k-) \geq 0, \tag{4.5}$$

$$[z(t)]'[y(t) - y(t_k-)] = 0. \tag{4.6}$$

We have thus demonstrated that (y, z) on $[t_k, t_{k+1})$ is the (constant) solution to

$$LCP_Q[z(t_{k-1}) + x(t_k) - x(t_{k-1})], \quad \text{for all } k = 1, 2, \ldots;$$

its existence and uniqueness is guaranteed by Proposition 4.3, in view of $e'[z(t_{k-1}) + x(t_k) - x(t_{k-1})] = e'x(t_k) > 0$.

4.1.A *Remark:* The above iterative procedure constitutes a proof that Ψ_Q is well-defined on the subset of step-functions in $D^J_>$. Since step-functions are dense in $D^J_>$, it is conceivable that some extension procedure might simplify our current proof of Theorem 2.5, and perhaps even strengthen it (for example, to Ψ_Q being well-defined on $D^J_>$, and uniformly continous on $C^J_>$). To this end, some inequalities in the spirit of (2.8)-(2.9) must be established, but we were unable to proceed along this avenue.

4.2 *Proof of Proposition 2.2:* To simplify the notation, underbars will be omitted from here on. Let y be the least element of $\pi_Q(x)$ and put $z = x + [I-Q]y$. Suppose, to the contrary, that for some $j = 1, \ldots, J$ and $t \geq 0$, both y_j increases at t and $z_j(t) > 0$. Two cases will be distinguished.

Case 1: y_j discontinuous at t. We shall show that it is possible to reduce the size of the jump at time t, without hurting feasibility. There exist $\epsilon, \delta > 0$, small enough, so that $y_j(t) - y_j(t-) \geq \epsilon$ and $z_j(s) \geq \epsilon$ for $t \leq s \leq t + \delta$. Define $\tilde{y}$ by

$$\tilde{y}_j(s) = \begin{cases} y_j(s), & 0 \leq s < t, \\ y_j(s) - \epsilon, & t \leq s < t + \delta, \\ y_j(s), & t + \delta \leq s < \infty, \end{cases}$$

and $\tilde{y}_i = y_i$ for all $i \neq j$. Then $\tilde{y} \in D^J_\uparrow$, because y jumps by at least ϵ at time t, and $x + [I-Q]\tilde{y} \geq 0$ by simple algebra. However, $\tilde{y} \leq y$ and $\tilde{y} \neq y$, which contradicts the efficiency of y.

Case 2: y continuous at t. The idea now is to freeze y for a short period beyond time t, and maintain feasibility. By the right-continuity of the functions involved, there exists $\epsilon, \delta > 0$ small enough, so that both $0 \leq y_j(s) - y_j(t) \leq \epsilon$, and $z_j(s) \geq \epsilon$, for $t \leq s < t + \delta$. Define $\tilde{y}$ by

$$\tilde{y}_j(s) = \begin{cases} y_j(s), & 0 \leq s < t, \\ y_j(t), & t \leq s < t + \delta, \\ y_j(s), & t + \delta \leq s < \infty, \end{cases}$$

and $\tilde{y}_i = y_i$ for $i \neq j$. Again $\tilde{y}$ is a feasible production plan, $\tilde{y} \leq y$ and $\tilde{y} \neq y$ (because $\tilde{y}_j(s) = y_j(t) < y_j(s)$ for $t \leq s < t+\delta$). We are faced with another contradiction, which completes the proof.

4.3 *Proof of Proposition 2.4:* The least element y solves $DCP_Q(x)$, as shown in Proposition 2.2. Assume that $\tilde{y}$ is another solution. We shall verify momentarily the following three claims:

4.3.A y and $\tilde{y}$ do coincide on $[0,\delta]$, for some $\delta > 0$;

4.3.B if $y(\tau) = \tilde{y}(\tau)$ at some $\tau \geq 0$, then

$$y(t) = \tilde{y}(t), \quad \text{on } t \in [\tau, \tau+\delta],$$

for some $\delta > 0$;

4.3.C if $y(t) = \tilde{y}(t)$ on $t \in [0,\tau)$, then also $y(\tau) = \tilde{y}(\tau)$.

Equipped with 4.3.A-4.3.C, let us define

$$\tau = \sup\{t \geq 0\colon\ y(s) = \tilde{y}(s) \text{ for all } 0 \leq s \leq t\}.$$

It follows from 4.3.A that $\tau \geq \delta$. Suppose that $\tau < \infty$. Then 4.3.C clearly prevails; hence, by 4.3.B, y and $\tilde{y}$ coincide somewhat beyond τ. This contradicts the definition of τ, it implies that $\tau = \infty$, and it will complete the proof, once 4.3.A-4.3.C are established.

The relation

$$e'z(t) = e'x(t) > 0, \quad \text{for all } t \geq 0,$$

will be repeatedly used during the proofs of 4.3.A-4.3.C; it is a consequence of Q being column-stochastic. First, let us verify 4.3.A. By $e'x(0) > 0$ and right-continuity, there exists an index, say $j = 1$, and $\delta > 0$, such that $y_1(t) = \tilde{y}_1(t) = 0$ for $t \in [0,\delta]$. Let $x_{\neq 1}$ denote the $(J-1)$-dimensional vector obtained from x by omitting its first coordinate, and similarly for $y_{\neq 1}, \tilde{y}_{\neq 1}$. Then both $y_{\neq 1}$ and $\tilde{y}_{\neq 1}$ solve $LCP_{Q_{\neq 1}}(x_{\neq 1})$ on $[0,\delta]$, where the matrix $Q_{\neq 1}$ is Q with first row and column omitted. Next, irreducibility of Q implies that $\sigma(Q_{\neq 1}) < 1$. The uniqueness guaranteed by Theorem 2.3 now establishes 4.3.A.

The proof of 4.3.B is reduced to that of 4.3.A. One simply shifts the origin to τ, through 2.8.D with $s = \tau$, while using the facts that $z(\tau) = \tilde{z}(\tau)$, and for $t \geq 0$

$$e'[z(\tau) + x(\tau+t) - x(\tau)] = e'x(\tau+t) > 0.$$

We are left with 4.3.C. Start with the observation that $z(\tau-) = \tilde{z}(\tau-)$, and denote their common value by ζ. Then both $[y(\tau) - y(\tau-)]$ and $[\tilde{y}(\tau) - \tilde{y}(\tau-)]$ solve $LCP_Q[\zeta + x(\tau) - x(\tau-)]$. But $e'[\zeta + x(\tau) - x(\tau-)] = e'x(\tau) > 0$, hence Proposition 4.3 applies. The conclusion is that $y(\tau) - y(\tau-) = \tilde{y}(\tau) - \tilde{y}(\tau-)$, and we are done.

4.4 This subsection is devoted to proving

Lemma 4.4. *Under the assumptions in Theorem 2.3 or Theorem 2.5, the process $\Psi_Q(x)$ is continuous if the data x is upper-semi-continuous.*

Proof: Let $y = \Psi_Q(x)$ and $z = \Phi_Q(x)$. By right-continuity and existence of left limits, we only need to show that $y(t) = y(t-)$ for all $t > 0$. The triplet (x,y,z) satisfies

$$\begin{aligned} &z(t) = [z(t-) + x(t) - x(t-)] + [I-Q][y(t) - y(t-)] \geq 0,\\ &[y(t) - y(t-)] \geq 0,\\ &z(t)'[y(t) - y(t-)] = 0, \end{aligned}$$

for all $t > 0$. Equivalently, $[y(t) - y(t-), z(t)]$ solves $LCP_Q[z(t-) + x(t) - x(t-)]$. By Theorem 10.2.15 in Berman and Plemmons [1979] if $\sigma(Q) < 1$, or by Proposition 4.3 if Q is irreducible and column-stochastic, this solution is unique. The upper-semi-continuity of x implies that

$$[z(t-) + x(t) - x(t-)] \geq z(t-) \geq 0;$$

therefore, $0 = y(t) - y(t-)$ is the unique solution.

4.5 *Proof of 2.8.G:* As in Theorem 1 of Harrison and Reiman [1981],

4.5.A $$y(t) = \sup_{0 \leq s \leq t} [Qy(s) - x(s)]^+,$$

from which one deduces that

$$z_i(t) = x_i(t) - \sum_{j=1}^{J} q_{ij} y_j(t) + \sup_{0 \leq s \leq t} \left[\sum_{j=1}^{J} q_{ij} y_j(s) - x_i(s) \right]^+.$$

Define $g_i(t) = \sum_j q_{ij} y_j(t)$ and $h_i(t) = g_i(t) - x_i(t)$; then

$$z_i(t) = \sup_{0 \leq u \leq t} [h_i(u)]^+ - h_i(t).$$

For $t \geq s$, we have

$$\begin{aligned} z_i(t) - z_i(s) + h_i(t) - h_i(s) &= \sup_{0 \le u \le t} [h_i(u)]^+ - \sup_{0 \le v \le s} [h_i(v)]^+ \\ &\le \sup_{s \le u \le t} [h_i(u) - h_i(s)] \\ &= \sup_{s \le u \le t} [g_i(u) - g_i(s) - (x_i(u) - x_i(s))] \\ &\le g_i(t) - g_i(s) + \sup_{s \le u \le t} [x_i(s) - x_i(u)]; \end{aligned}$$

here, the first inequality is based on $\alpha^+ - \beta^+ \le (\alpha - \beta)^+$, and the second on the monotonicity of g_i. We proceed with

$$\begin{aligned} z_i(t) - z_i(s) \le x_i(t) - x_i(s) + \sup_{s \le u \le t} [x_i(s) - x_i(u)] \\ \le \sup_{s \le u \le t} [x_i(t) - x_i(u)]. \end{aligned}$$

Adding the positive-sign function, which is legitimate, yields (2.8); (2.9) immediately follows suit.

4.5.B *Remark:* The inequality (2.7) was proved by Ludo Van der Heiden (unpublished), for the open-like non-Leontief economies described in Section 5. In its present form, (2.7) is a consequence of 4.5.A.

4.6 A first step in proving Theorem 2.5 is

Lemma 4.5. *Let Q be column-stochastic and irreducible. Then Ψ_Q is well defined on $C_1^J[0,T]$, for all $T > 0$.*

To prove Lemma 4.5, we construct an element in $\pi_Q(x)$ as follows. Any solution z of 2.4.A-2.4.C satisfies $e'z(t) = e'x(t) = 1$, and $z(t) \ge 0$, for all $t \ge 0$. Consequently, there must always be at least one component of $z(t)$ whose value is greater than or equal to $1/J$. Suppose, to start with, that $z_1(0) \ge 1/J$. As in the proof of 4.3.A, one sets $y_1(t) = 0$, then solves a $(J-1)$-dimensional DCP associated with an open system. Theorem 2.3 guarantees the existence and uniqueness of such a solution, until the first time $\tau > 0$ that $z_1(\tau) = 0$. (If $\tau = \infty$, we are done.) At that time, $z_1(\tau) = 0$, hence there exists another component of $z(\tau)$ that is greater than or equal to $1/J$. We freeze the corresponding y-component, thus switching to another $(J-1)$-dimensional open system, and so on. Iterating this way, one constructs, piece by piece, a solution for all times $T > 0$. The pieces are formally concatenated with the help of 2.8.E, and uniqueness of the outcome is guaranteed by Proposition 2.4. We now proceed with the details.

Proof of Lemma 4.5: For any $a \in \mathbb{R}^J$, denote by $a_{\neq k}$ the $(J-1)$-dimensional vector obtained from a by omitting its k-th component. Since $z(0) = x(0) \ge 0$ and $e'z(0) = e'x(0) = 1$, there exists an index j such that $z_j(0) \ge 1/J$. Without loss of generality, assume that $z_1(0) \ge 1/J$. Represent

$$Q = \begin{pmatrix} q_{11} & r' \\ c & P \end{pmatrix}$$

where P is a $(J-1) \times (J-1)$ matrix, and r, c are $(J-1)$-dimensional vectors.

Since Q is an irreducible stochastic matrix, P is substochastic with $\sigma(P) < 1$ and $q_{11} < 1$ (Corollary 2.1.6 and Lemma 6.2.1 in Berman and Plemmons [1979].) To simplify notation, let us assume that $q_{jj} = 0$, $j = 1, \ldots, J$.

By letting

$$x = \begin{pmatrix} x_1 \\ x_{\neq 1} \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_{\neq 1} \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} z_1 \\ z_{\neq 1} \end{pmatrix},$$

we can rewrite 2.4.A as

$$z_1(t) = x_1(t) + y_1(t) - r' y_{\neq 1}(t), \tag{4.7}$$

$$z_{\neq 1}(t) = x_{\neq 1}(t) - c\, y_1(t) + [I - P] y_{\neq 1}(t). \tag{4.8}$$

Note that $y_1(0) = 0$. Applying Theorem 2.3 yields a unique solution pair $(y_{\neq 1}, z_{\neq 1})$ that satisfies

$$z_{\neq 1}(t) = x_{\neq 1}(t) - c\, y_1(0) + [I - P] y_{\neq 1}(t) \ge 0, \quad t \ge 0, \tag{4.9}$$

$$y_{\neq 1} \text{ is non-decreasing with } y_{\neq 1}(0) = 0, \tag{4.10}$$

$$y_i \text{ increases at } t \ge 0 \text{ only if } z_i(t) = 0, \quad i = 2, \ldots, J. \tag{4.11}$$

Now fix δ, $0 \le \delta < 1/J$, throughout the proof. (For our present purposes, it suffices to work with $\delta = 0$. Later however, during the proof of Lemma 4.6, $\delta > 0$ will be required.) Let $T_1 = \inf\{t \ge 0\colon\ e'x(t) - e'z_{\neq 1}(t) \le \delta\}$. It $T_1 = \infty$ then we are done, so assume that

$T_1 < \infty$. As $z_{\neq 1}(t)$ and $x(t)$ are continuous, and $e'x(0) - e'z_{\neq 1}(0) = z_1(0) \geq 1/J > \delta$, it must be that $T_1 > 0$ and $e'z_{\neq 1}(t) \leq e'x(t)$ for $t \in [0, T_1]$. It follows that $(y(t), z(t))$, where

$$y(t) = \begin{pmatrix} y_1(0) \\ y_{\neq 1}(t) \end{pmatrix} \quad \text{and} \quad z(t) = \begin{pmatrix} e'x(t) - e'z_{\neq 1}(t) \\ z_{\neq 1}(t) \end{pmatrix},$$

is a solution pair to 2.4.A-2.4.C, for all $t \in [0, T_1]$.

Replace now $x(t)$ by $z(T_1) + x(T_1 + t) - x(T_1)$. The above procedure can be used to identify a solution-pair for all $t \in [0, T_2]$ $(T_2 > 0)$. Applying 2.8.E, we obtain a solution pair (y, z) to 2.4.A-2.4.C, for $t \in [0, T_1 + T_2]$. Iterating with this procedure, we come up with a solution (y, z) to 2.4.A-2.4.C, for all $t \in [0, \sum_{k=1}^{N} T_k]$, $N \geq 1$.

We now claim that for any $T > 0$, $\sum_{k=1}^{N} T_k \geq T$ for some finite N. This, of course, will establish existence. Suppose that for some finite $T > 0$, $S_N = \sum_{k=1}^{N} T_k \leq T$ for all $N \geq 1$. Without loss of generality, assume that the first equation in 2.4.A was the one to be ignored, in the $(k+1)$-st iteration. Then by the definition of T_k, one can show that

$$e'z_{\neq 1}(S_{k+1}) - e'z_{\neq 1}(S_k) \geq \frac{1}{J} - \delta + e'x(S_{k+1}) - e'x(S_k);$$

hence,

$$\max_j \sup_{S_k \leq s \leq t \leq S_{k+1}} [z_j(t) - z_j(s)]$$
$$\geq \frac{1}{J}\left[\frac{1}{J} - \delta\right] - \max_j \sup_{S_k \leq s \leq t \leq S_{k+1}} |x_j(t) - x_j(s)|, \qquad \text{for all } k \geq 1.$$

By 2.8.G, we have

$$\max_j \sup_{S_k \leq s \leq t \leq S_{k+1}} |x_j(t) - x_j(s)| \geq \frac{1}{2J}\left[\frac{1}{J} - \delta\right] > 0, \tag{4.12}$$

for all $k \geq 1$. However, $x(t)$ is uniformly continuous on $[0, T]$. Hence, there exists a $\eta > 0$ such that

$$|x_j(t) - x_j(s)| < \frac{1}{2J}\left[\frac{1}{J} - \delta\right],$$

for all $s, t \in [0, T]$ with $|s - t| < \eta$, and $j = 1, \ldots, J$.

The inequality (4.12), and the assumption $S_k < T$ for all $k \geq 1$, imply that $T_k \geq \eta$ for all $k \geq 1$. This contradics the finiteness of T, and establishes Lemma 4.5.

4.7 Next we show that, under the assumption of Lemma 4.5, the regulator mapping is continuous.

Lemma 4.6. *The mapping Ψ_Q is continuous on $C_1^J[0, T]$, for any $T > 0$.*

Proof: Throughout the proof, write Ψ and Φ instead of Ψ_Q and Φ_Q. For $j = 1, \ldots, J$, let Q_j be the j-th principal minor of the matrix Q, namely the $(J-1)\times(J-1)$ matrix obtained from Q by deleting its j-th row and column. Since Q is an irreducible column-stochastic matrix, all Q_j, $j = 1, \ldots, J$, must be substochastic matrices with $\sigma(Q_j) < 1$, $j = 1, \ldots, J$. Simplify Ψ_{Q_j} and Φ_{Q_j} to Ψ_j and Φ_j, for convenience. By Lipshitz continuity (Remark 2.5.A), there exists $\theta > 0$ such that

$$\|\Psi_j(x^1) - \Psi_j(x^2)\| \leq \theta \|x^1 - x^2\|, \qquad \text{and}$$
$$\|\Phi_j(x^1) - \Phi_j(x^2)\| \leq \theta \|x^1 - x^2\|, \qquad j = 1, \ldots, J,$$

for all $x^1, x^2 \in D_0^{J-1}$.

Fix $T > 0$ and $x \in C_1^J$. Pick δ with $0 < \delta < 1/J$, construct y and z from x, as in the proof of Lemma 4.7, and finally recall that $T_k > 0$, $k = 1, \ldots, N$, and assume that $\sum_{k=1}^{N} T_k = T$. Put $S_k = \sum_{i=1}^{k} T_i$, $k = 1, \ldots, N$, and $S_0 = 0$.

For any $\epsilon > 0$, we choose $\eta > 0$ such that

$$\frac{(1 + J\theta)^N - 1}{J}\eta < \epsilon, \qquad \text{and}$$
$$\{J + (J-1)\theta + (J-1)\theta\left[(1 + J\theta)^{N-1} - 1\right]\}\eta < \delta.$$

We would like to prove that

$$\|y^* - y\| < \epsilon, \tag{4.13}$$

for $\|x^* - x\| < \eta$ and x^* in the domain of Ψ; here $y = \Psi(x)$ and $y^* = \Psi(x^*)$.

Denote by $\tau(k+1)$ the equation that was eliminated in the construction of (y, z), during the time interval $[S_k, S_{k+1}]$. We will prove, by induction, that

$$\|y^* - y\|_{S_k} < \frac{(1 + J\theta)^k - 1}{J}\eta, \qquad k = 1, \ldots, N, \tag{4.14}$$

which establishes (4.13).

For $k = 1$,

$$\begin{aligned} y_{\tau(1)}(t) &= 0, \\ y_{\neq\tau(1)}(t) &= \Psi_{\tau(1)}(x_{\neq\tau(1)})(t), \end{aligned} \qquad \text{for} \quad 0 \leq t \leq S_1. \tag{4.15}$$

In order to show that

$$\begin{aligned} &y^*_{\tau(1)}(t) = 0, \\ &y^*_{\neq\tau(1)}(t) = \Psi_{\tau(1)}(x^*_{\neq\tau(1)})(t), \end{aligned} \qquad \text{for} \quad 0 \le t \le S_1, \tag{4.16}$$

it suffices to show that

$$z^*_{\tau(1)}(t) > 0 \qquad \text{for} \quad 0 \le t \le S_1. \tag{4.17}$$

For $0 \le t \le S_1$, since $Z_{\tau(1)}(t) \ge \delta$ and

$$\begin{aligned} &\|[e'x^* - e'\Phi_{\tau(1)}(x^*_{\neq\tau(1)})] - z_{\tau(1)}\|_{S_1} \\ =&\|[e'x^* - e'x] - [e'\Phi_{\tau(1)}(x^*_{\neq\tau(1)}) - e'\Phi_{\tau(1)}(x_{\neq\tau(1)})]\|_{S_1} \\ \le& J\|x^* - x\|_{S_1} + (J-1)\theta\|x^* - x\|_{S_1} \\ <&(J + (J-1)\theta)\eta < \delta, \end{aligned}$$

we have

$$e'x^*(t) - e'\Phi_{\tau(1)}(x^*_{\neq\tau(1)})(t) > z_{\tau(1)} - \delta \ge 0. \tag{4.18}$$

By uniqueness,

$$\begin{aligned} &z^*_{\tau(1)}(t) = e'x^*(t) - e'\Phi_{\tau(1)}(x^*_{\neq\tau(1)})(t) \\ &z^*_{\neq\tau(1)}(t) = \Phi_{\tau(1)}(x^*_{\neq\tau(1)})(t), \end{aligned} \qquad \text{for} \quad 0 \le t \le S_1;$$

this, combined with (4.18), proves (4.17).

The equalities (4.15) and (4.16) imply that

$$\begin{aligned} \|y^* - y\|_{S_1} &= \|\Psi_{\tau(1)}(x^*_{\neq\tau(1)}) - \Psi_{\tau(1)}(x_{\neq\tau(1)})\|_{S_1} \\ &\le \theta\|x^*_{\neq\tau(1)} - x_{\neq\tau(1)}\|_{S_1} < \theta\eta, \end{aligned}$$

proving (4.14) for $k = 1$.

Now assume that (4.14) holds for $k < N$, and let us prove it for $k+1$. From the construction of y and z,

$$\begin{aligned} &z_{\tau(k+1)}(t) \ge \delta, \\ &y_{\tau(k+1)}(t) = y_{\tau(k+1)}(S_k), \qquad \text{and} \\ &y_{\neq\tau(k+1)}(t) = y_{\neq\tau(k+1)}(S_k) + \Psi_{\tau(k+1)}(\hat{x}_{\neq\tau(k+1)})(t - S_k), \end{aligned}$$

for $S_k \le t \le S_{k+1}$; here

$$\hat{x}(t) = x(t + S_k) - x(S_k) + z(S_k) = x(t + S_k) + [I - Q]y(S_k),$$

for $0 \le t \le T_{k+1}$. We claim that

$$\begin{aligned} &y^*_{\tau(k+1)}(t) = y^*_{\tau(k+1)}(S_k), \qquad \text{and} \\ &y^*_{\neq\tau(k+1)}(t) = y^*_{\neq\tau(k+1)}(S_k) + \Psi_{\tau(k+1)}(\hat{x}^*_{\neq\tau(k+1)})(t - S_k), \end{aligned} \tag{4.19}$$

for $S_k \le t \le S_{k+1}$, where for $0 \le t \le T_{k+1}$,

$$\hat{x}^*(t) = x^*(t + S_k) - x^*(S_k) + z^*(S_k) = x^*(t + S_k) + [I - Q]y^*(S_k).$$

Assume momentarily that (4.19) holds. Then

$$\begin{aligned} &\|y^* - y\|_{[S_k, S_{k+1}]} \le \|y^*_{\tau(k+1)}(S_k) - y_{\tau(k+1)}(S_k)\| \\ &+ \|\Psi_{\tau(k+1)}(\hat{x}^*_{\neq\tau(k+1)})(\cdot - S_k) - \Psi_{\tau(k+1)}(\hat{x}_{\neq\tau(k+1)})(\cdot - S_k)\|_{[S_k, S_{k+1}]} \\ &\le \|y^* - y\|_{S_k} + \theta\|\hat{x}^* - \hat{x}\|_{T_{k+1}} \\ &\le \|y^* - y\|_{S_k} + \theta\big[\|x^*(\cdot + S_k) - x(\cdot + S_k)\|_{T_{k+1}} \\ &\qquad\qquad + \|[I - Q][y^*(S_k) - y(S_k)]\|\big] \\ &\le (1 + J\theta)\|y^* - y\|_{S_k} + \theta\eta \\ &< (1 + J\theta)\frac{(1 + J\theta)^k - 1}{J}\eta + \theta\eta = \frac{(1 + J\theta)^{k+1} - 1}{J}\eta. \end{aligned}$$

The proof will be completed once (4.19) is verified. As we did in the case of $k = 1$, it suffices to show that

$$z^*_{\tau(k+1)}(t) > 0 \qquad \text{for} \quad S_k \le t \le S_{k+1}. \tag{4.20}$$

Note that

$$\begin{aligned} &z_{\neq\tau(k+1)}(t) = \Phi_{\tau(k+1)}(\hat{x}_{\neq\tau(k+1)})(t - S_k), \qquad \text{and} \\ &z_{\tau(k+1)}(t) = e'x(t) - e'z_{\neq\tau(k+1)}(t) \end{aligned}$$

for $S_k \le t \le S_{k+1}$. Then

$$\begin{aligned} &\|[e'x^* - e'\Phi_{\tau(k+1)}(\hat{x}^*_{\neq\tau(k+1)})(\cdot - S_k)] - z_{\tau(k+1)}\|_{[S_k, S_{k+1}]} \\ \le& J\|x^* - x\|_{[S_k, S_{k+1}]} + (J-1)\theta\|\hat{x}^* - \hat{x}\|_{T_{k+1}} \\ <& J\eta + (J-1)\theta[\eta + J\|y^*(S_k) - y(S_k)\|] \\ <& \left\{[J + (J-1)\theta] + (J-1)\theta[(1 + J\theta)^k - 1]\right\}\eta < \delta; \end{aligned}$$

hence,

$$e'x^*(t) - e'\Phi_{\tau(k+1)}(\hat{x}^*_{\neq\tau(k+1)})(t - S_k) > z_{\tau(k+1)} - \delta \geq 0.$$

By uniqueness,

$$z^*_{\tau(k+1)}(t) = e'x^*(t) - e'\Phi_{\tau(k+1)}(\hat{x}^*_{\neq\tau(1)})(t - S_k) > 0$$

for $S_k \leq t \leq S_{k+1}$. The proof is now complete.

4.8 *Proof of Theorem 2.5:* First it is shown that Ψ_Q is well defined on $D_1^{J,f}[0,T]$. Any $x \in D_1^{J,f}[0,T]$ has only a finite number of discontinuity points on $[0,T]$. To simplify the presentation, let us assume that x has a single jump at $s \in (0,T]$ ($s > 0$ because x is right continuous). By Lemma 4.7, we can find (y,z) satisfying 2.4.A-2.4.C with x replaced by x^* for $t \in [0,s]$, where $x^*(t) = x(t)$ for $t \in [0,s)$ and $x^*(s) = x(s-)$. It is clear that this same pair (y,z) satisfies 2.4.A-2.4.C with x, for $t \in [0,s)$.

Define $(y(s), z(s))$ by

$$\begin{aligned} &z(s) = z(s-) + x(s) - x(s-) + [I - Q][y(s) - y(s-)] \geq 0, \\ &y(s) - y(s-) \geq 0, \\ &z(s)'[y(s) - y(s-)] = 0, \end{aligned}$$

where existence and uniqueness of $(y(s), z(s))$ is guaranteed by Proposition 4.3. Again, applying Lemma 4.5, we can find (y^*, z^*) satisfying 2.4.A-2.4.C with x replaced by $z(s) + x(s+t) - x(s)$ for $t \in [0, T-s]$. Now, by 2.8.E, construct (y,z) satisfying 2.4.A-2.4.C on $[0,T]$. The uniqueness follows from Proposition 2.4.

All our arguments, so far, are applicable if, instead of $e'x \equiv 1$, we allow $e'x \geq 1$. By rescaling x (property 2.8.C), we deduce that Ψ_Q is actually well defined on $D^J_{+\epsilon}[0,t]$, for all $\epsilon > 0$ and $t > 0$.

We are now capable of showing that Ψ_Q is well defined on $D^J_{>}$. Fix $x \in D^J_{>}$. For any integer n, $x \in D^J_\epsilon[0,n]$ for some $\epsilon > 0$; therefore, there exists an unique pair (y^n, z^n) satisfying 2.4.A-2.4.C for $t \in [0,n]$. Uniqueness implies that $(y^m(t), z^m(t)) = (y^n(t), z^n(t))$ for $t \in [0,n]$ if $m \geq n$. Hence, (y^n, z^n) is extendable consistently, as $n \to \infty$, to a unique solution for 2.4.A-2.4.C, on $[0,\infty)$.

Finally, we check that Ψ_Q is continuous on $C^J_{>}$. It suffices to prove continuity on $C^J_{+\epsilon}[0,t]$, for any fixed $\epsilon > 0$ and $t > 0$; this is a consequence of Lemma 4.6 and property 2.8.C.

5. Closing Remarks and Further Research

Our main goal has been to record some mathematical prerequisites for CMa, CMb and Harrison, Williams and Chen [1989]. This could have also been achieved through the fluid-flow models in CMa, or a mere theorem/proof presentation. The reader, hopefully, agrees that the choice of the "Leontief route" has some merit.

Additional work can be attempted in the following directions:

The Regulator Mapping: We have not identified the exact domain of the regulator mapping in the case where Q is irreducible with $\sigma(Q) = 1$. Such an identification should be also supplemented with a proof that the regulator is continuous on its domain. One could attempt to approach these problems as follows. Consider the least-element mapping, which is a *selection* of the *correspondence* $x \to \pi_Q(x)$; it maps x to the least element in $\pi_Q(x)$. Continuity of the least element mapping would follow from lower-semi-continuity of the correspondence.

Open Systems: If $\sigma(Q) < 1$ then $LCP_Q(\xi)$ in 3.9.C-3.9.E has a unique solution, and $\zeta \neq 0$ turns out to be possible. The set of sectors γ with $\zeta_\gamma > 0$ (hence $\eta_\gamma = 0$, by 3.9.E), are those sectors in which inventory levels explode due to excess supply. Such open models can be accomodated within our framework without too much effort, as in Section 6 of CMb.

Non-Leontief Economies: A natural extension of the dynamic complementarity problem 2.4.A-2.4.C is to allow the Leontief matrix $[I - Q]$ to be a general input-output matrix, say R (by replacing the identity matrix in $[I - Q]$ with an arbitrary nonnegative matrix P, so that $R = P - Q$). Mandelbaum [1989] advocates that the resulting DCP provides an interesting object of study: it is known to posses a solution for every netput x if and only if R is a strictly-semi-monotone matrix (SSM matrix, also known as a completely-Q matrix; see, for example, Berman and Plemmons [1979]). An outstanding open problem is the characterization of the SSM matrices for which the solution to the associated DCP is also unique. (Being a P matrix is not sufficient, as might be wrongly anticipated from the classical theory of LCP; see Mandelbaum [1989], Subsection 3.3 for an example.) The key observation of the relation between the regulator mapping and LCP-theory is due to Mike Harrison. This observation stimulated the connection of SSM matrices to DCP-theory, first by Ludo Van der Heyden in an unpublished work, and then by Reiman and Williams [1988]. Faced with the possibility of non-uniqueness, measurability issues must be carefully dealt with. (See Bernard and Kahrroubi [1989], which was brought to our attention by Ruth Williams.)

DCP can be also identified as an LCP in an infinite dimensional space (Borwein and Dempster [1989]), a convex process (Rockafellar [1974], especially Section 10), or a generalized differential inclusion (Aubin and Cellina [1984]). It remains to check whether such identifications can simplify some of our tedious arguments in Section 4.

Economics: Economists typically view Leontief systems as macroscopic models. In our case, a macroscopic description is the deterministic linear model in Theorem 3.3. It provides an aggregated version of the microscopic stochastic model 3.4.A-3.4.C. The diffusion model, emerging from Theorem 3.4, is intermediate or mesoscopic; it bridges between the detailed microscopic and crude macroscopic specifications.

Some details could be added to the microscopic model. For example, one could incorporate inventory costs or prices of finished goods, and then optimize. (The least production plan need no longer be optimal.) The macroscopic and mesoscopic approximations of the optimization model give rise, respectively, to deterministic control and singular control problems. Establishing turnpike theorems that connect the "micro" with the "macro" and "meso" seems like a worthwhile endeavor.

Appendix: Notation and Conventions

Vectors and Matrices: Vectors are understood to be column vectors. The transpose of a vector or a matrix is obtained by adding to it a "prime". The J-dimensional Euclidean space is denoted by $\mathbb{R}^J$, its nonnegative orthant by $\mathbb{R}^J_+$, and the unit-vectors in $\mathbb{R}^J$ by $e^j, j = 1, \ldots, J$. When the letter e represents a vector, it is the vector of ones.

Let $x = (x_1, \ldots, x_J)' \in \mathbb{R}^J$, and $\alpha \subseteq \{1, \ldots, J\}$. Then the scalar $|\alpha|$ denotes the cardinality of α, and the vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ is the restriction of x to its coordinates with indices in α. Similarly, the matrix $P_{\alpha\beta}$ is the submatrix of a matrix P obtained by choosing the elements with row-indices in α and column-indices in β; $P_{\alpha\alpha}$ will be abbreviated to P_α. Vector (in)equalities are to be interpreted component-wise. In particular, $x > y$ means that each coordinate of x is strictly greater than the corresponding y coordinate.

The vector $|x| \in R^J$ stands for $|x| = (|x_1|, \ldots, |x_J|)'$, while $||x|| = max_{1 \le j \le J}|x_j|$.

The spectral radius of a square matrix A is denoted by $\sigma(A)$.

Scalar Functions: The positive-part of a scalar r is denoted by $r^+ = max\{0, r\}$, the negative part by $r^- = max\{0, -r\}$ and the integer-part (the largest integer not exceeding r) by $\lfloor r \rfloor$. The indicator function of a set S is the function 1_S that equals 1 on S and 0 otherwise. A real-valued nondecreasing function $y(t)$ has a point of increase (or simply increases) at $t \ge 0$ if

$$y(t + \epsilon) > y(t-) \quad \text{for all } \epsilon > 0,$$

with the convention $y(0-) = y(0)$. (Equivalently, y does *not* increase at $t \ge 0$ if it is both continuous and right-flat at t.)

Vector Functions: An operation of a scalar function on a vector is to be interpreted coordinate-wise. A vector-valued function is nondecreasing (nonincreasing) when all its components are nondecreasing (nonincreasing).

Function Spaces: Denote by $D^J[0, \infty)$ the space of J-dimensional RCLL (Right-Continuous with Left Limits) functions, namely the $\mathbb{R}^J$-valued functions on $[0, \infty)$ that are right continuous on $[0, \infty)$ with finite left limits on $(0, \infty)$. We distinguish in $D^J[0, \infty)$ the following subsets:

$$D_0^J[0, \infty) = \{x \in D^J[0, \infty) : x(0) \ge 0\},$$

$$D_=^J[0, \infty) = \{x \in D_0^J[0, \infty) : e'x(t) \equiv e'x(0) > 0 \text{ for all } t \ge 0\},$$

$$D_1^J[0, \infty) = \{x \in D_0^J[0, \infty) : e'x(t) = 1 \text{ for all } t \ge 0\},$$

$$D_+^J[0, \infty) = \{x \in D^J[0, \infty) : x(t) \ge 0 \text{ for all } \ t \ge 0\},$$

$$D_\uparrow^J[0, \infty) = \{x \in D^J[0, \infty) : x(0) = 0,\ x(t) \text{ is nondecreasing in } t \ge 0\};$$

$C^J[0, \infty), C_0^J[0, \infty), C_=^J[0, \infty), \ldots$ stand for the set of *continuous* functions in the respective D^J-spaces, $B_\bullet^J[0, \infty)$ consists of the functions in $D_\bullet^J[0, \infty)$ which have bounded-variation over finite intervals (locally of bounded variation); $D_\bullet^J[0, T)$ denotes the restrictions of functions in $D_\bullet^J[0, \infty)$ to $[0, T)$, $T > 0$, and $C_\bullet^J[0, T]$ are to be interpreted similarly. To simplify notation, $D_\bullet^J[0, \infty)$ will be abbreviated to $D_\bullet^J$, as will $C_\bullet^J[0, \infty)$ and $B_\bullet^J[0, \infty)$. Some additional subsets of D^J will be introduced when they arise.

For any $x \in D^J$ and $t > s > 0$, the uniform norm of x on the interval $[s,t]$ is the (necessarily finite) quantity

$$\|x\|_{[s,t]} = \sup_{s \leq u \leq t} \|x(u)\|,$$

with $\|x\|_{[0,t]}$ abbreviated to $\|x\|_t$.

Convergence: A sequence of functions in D^J converges "*u.o.c.*" if it converges uniformly on compact subsets of the semi-closed interval $[0,\infty)$. The sequence converges "*u.o.c.* in $t > 0$" if the convergence is uniform on compact subsets of the open interval $(0,\infty)$.

Convergence in distribution and weak convergence are used interchangeably, and both are denoted by $\xrightarrow{d}$. Let $\{X_n\}$ be a sequence of stochastic processes with sample paths in D^J. Billingsley [1968] is the standard reference for results on the weak convergence of $\{X_n\}$, when $J = 1$ and time is restricted to $[0,1]$. Extensions to $J \geq 1$ and to the time set $[0,\infty)$ have also become standard (see, for example, Whitt [1980] or Chapter 3 in Ethier and Kurtz [1986]). We further use "weak convergence on $t > 0$" to denote weak convergence in $D^J(0,\infty)$, ($t = 0$ excluded). Whitt [1980] provides the modifications required to account for the exclusion of the origin (Resnik [1987] is an alternative text-book source).

A final comment is that, in this paper, weak convergence is exclusively in D^J. However, all the stochastic processes that arise as weak limits have continuous sample paths, at least on $(0,\infty)$. Consequently, *u.o.c.* convergence of RCLL functions suffices for our purposes, and there is no need to either introduce, discuss or use convergence with respect to any of Skorohod's metrics (see, for example, Pollard [1984] who considers $[0,\infty)$).

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