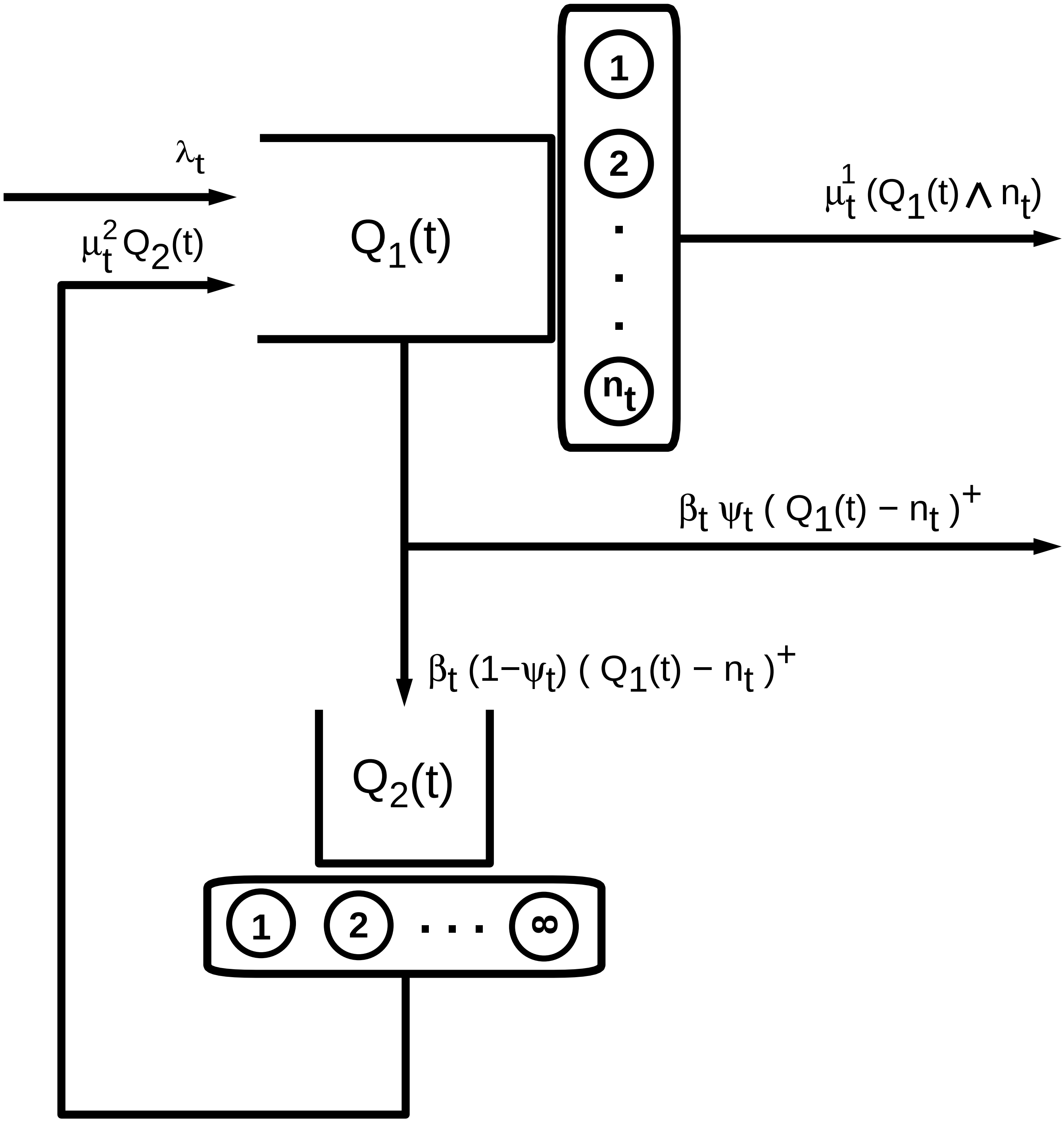
Back to the Multiserver Queue with Abandonment and Retrials



Sample Path Construction of a Multiserver Queue with Abandonment and Retrials

$$
\begin{aligned}
Q_1(t) &= Q_1(0) + A^a\left(\int_0^t \lambda_s ds\right) \\
&+ A^c_{21}\left(\int_0^t Q_2(s)\mu_s^2 ds\right) - A^c\left(\int_0^t (Q_1(s)\wedge n_s)\mu_s^1 ds\right) \\
&- A^b_{12}\left(\int_0^t (Q_1(s)-n_s)^+\beta_s(1-\psi_s)ds\right) \\
&- A^b\left(\int_0^t (Q_1(s)-n_s)^+\beta_s\psi_s ds\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2(t) &= \\
&Q_2(0) + A^b_{12}\left(\int_0^t (Q_1(s)-n_s)^+\beta_s(1-\psi_s)ds\right) \\
&- A^c_{21}\left(\int_0^t Q_2(s)\mu_s^2 ds\right).
\end{aligned}
$$

$A_{\cdot} \stackrel{d}{=}$ Poisson(1), independent.

Fluid Limit for the Multiserver Queue with Abandonment and Retrials (2 O.D.E.'s)

$$\frac{d}{dt}\, Q_1^{(0)}(t) \;=\; \lambda_t + \mu_t^2\, Q_2^{(0)}(t) - \mu_t^1 \left(Q_1^{(0)}(t) \wedge n_t\right) - \beta_t \left(Q_1^{(0)}(t) - n_t\right)^+$$

and

$$\frac{d}{dt}\, Q_2^{(0)}(t) = \beta_t(1-\psi_t) \left(Q_1^{(0)}(t) - n_t\right)^+ - \mu_t^2\, Q_2^{(0)}(t)\,.$$

Can be solved numerically (forward Euler) in a spreadsheet.

Diffusion Moments for the Multiserver Queue with Abandonment and Retrials

Let $E_1(t) = E\left[Q_1^{(1)}(t)\right]$, $E_2(t) = E\left[Q_2^{(1)}(t)\right]$.

Assume the set $\left\{t \,\middle|\, Q_1^{(0)}(t) = n_t\right\}$ has Lebesque measure zero.

Then

$$\begin{aligned} \frac{d}{dt} E_1(t) &= -\left(\mu_t^1 \mathbf{1}_{\{Q_1^{(0)}(t) \le n_t\}} + \beta_t \mathbf{1}_{\{Q_1^{(0)}(t) > n_t\}}\right) E_1(t) \\ &\quad + \mu_t^2 E_2(t) \end{aligned}$$

and

$$\frac{d}{dt} E_2(t) = \beta_t (1 - \psi_t) E_1(t) \mathbf{1}_{\{Q_1^{(0)}(t) \ge n_t\}} - \mu_t^2 E_2(t).$$

More Diffusion Moments

(A Grand Total of 7 O.D.E.'s)

Let $V_1(t) = \text{Var}\left[Q_1^{(1)}(t)\right]$, $V_2(t) = \text{Var}\left[Q_2^{(1)}(t)\right]$,

and $C(t) = \text{Cov}\left[Q_1^{(1)}(t), Q_1^{(1)}(t)\right]$. Then

$$\begin{aligned}
\frac{d}{dt} V_1(t) &= -2\left(\beta_t \mathbf{1}_{\{Q_1^{(0)}(t) > n_t\}} + \mu_t^1 \mathbf{1}_{\{Q_1^{(0)}(t) \le n_t\}}\right) V_1(t) \\
&\quad + \lambda_t + \beta_t \left(Q_1^{(0)}(t) - n_t\right)^+ + \mu_t^1 \left(Q_1^{(0)}(t) \wedge n_t\right) \\
&\quad + \mu_t^2 Q_2^{(0)}(t), \\
\frac{d}{dt} V_2(t) &= -2\mu_t^2 V_2(t) + \beta_t(1-\psi_t)\left(Q_1^{(0)}(t) - n_t\right)^+ \\
&\quad + \mu_t^2 Q_2^{(0)}(t) + 2\beta_t(1-\psi_t) C(t) \mathbf{1}_{\{Q_1^{(0)}(t) \ge n_t\}},
\end{aligned}$$

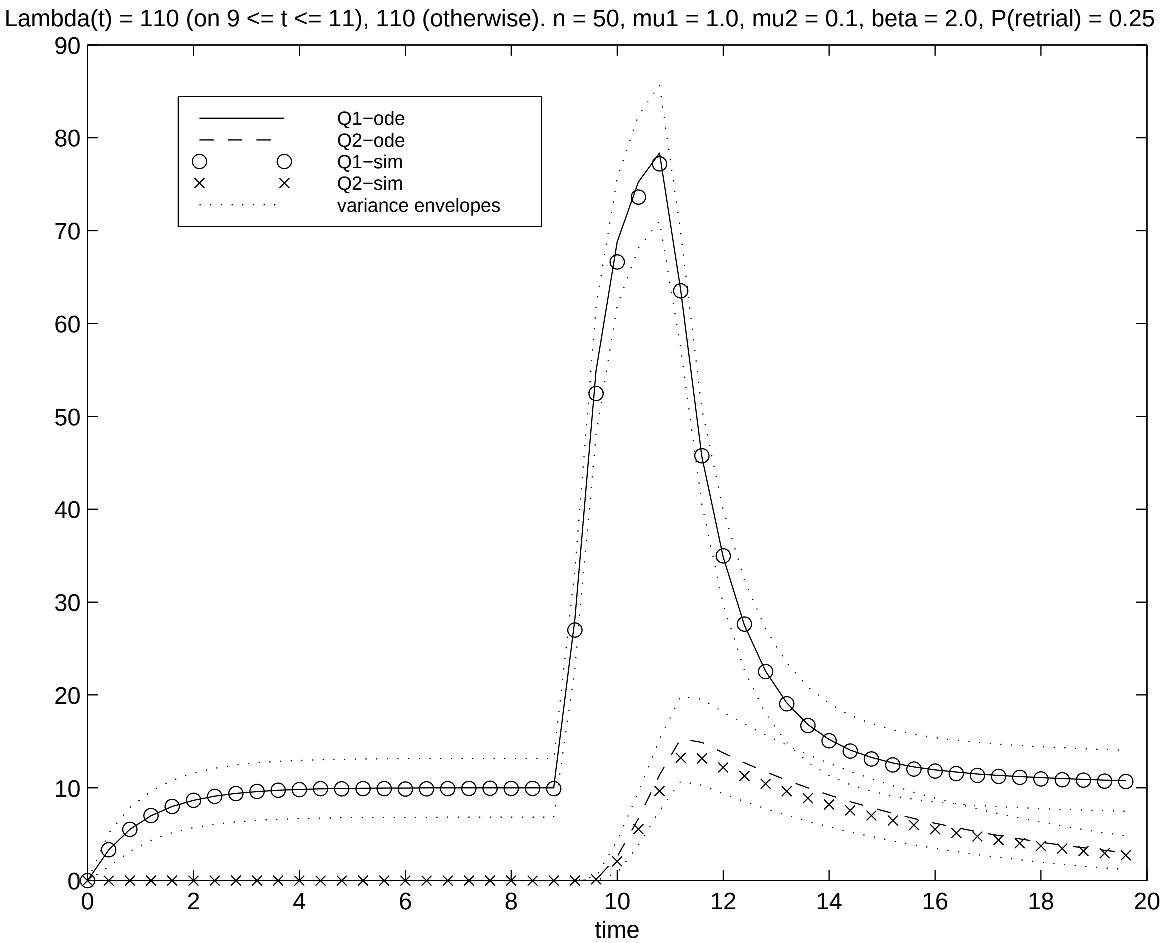
and

$$\begin{aligned}
\frac{d}{dt} C(t) &= -\left(\beta_t \mathbf{1}_{\{Q_1^{(0)}(t) \ge n_t\}} + \mu_t^1 \mathbf{1}_{\{Q_1^{(0)}(t) < n_t\}}\right) C(t) \\
&\quad + \mu_t^2 (V_2(t) - C(t)) - \beta_t(1-\psi_t)\left(Q_1^{(0)}(t) - n_t\right) \\
&\quad - \mu_t^2 Q_2^{(0)}(t).
\end{aligned}$$

Example: Spiked Arrival Rate:

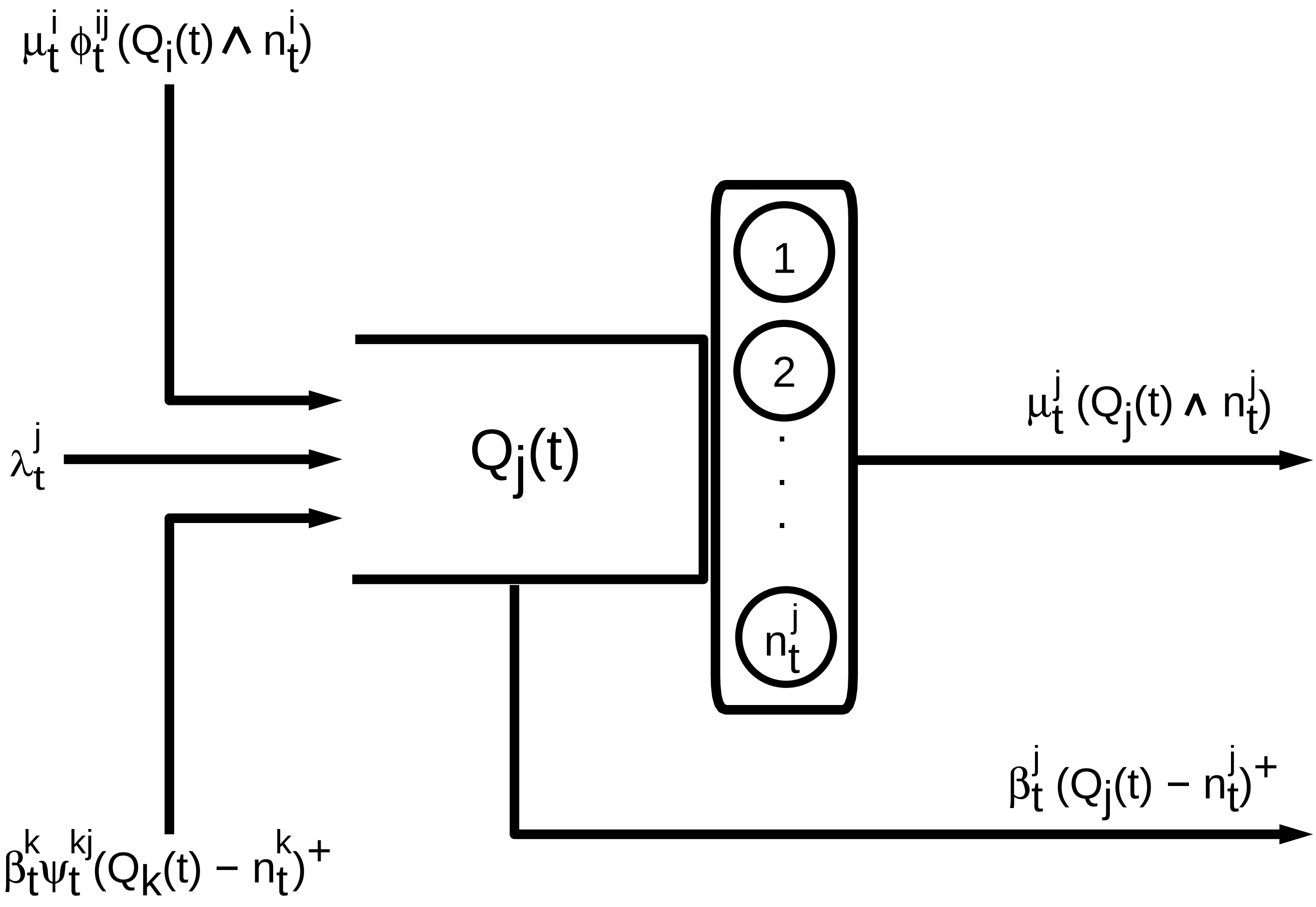
$\lambda(t) = 110$, if $9 \leq t \leq 11$ otherwise $\lambda(t) = 10$,

$\mu_1 = 1.0$, $\mu_2 = 0.1$, $\beta = 2.0$, $n = 50$, $\psi = 0.25$



Theory Generalizes to

Jackson Networks with Abandonment



Further generalizations: **Pre-Emptive Priorities**

Definition of Lipschitz Functions

A function $f : V \to \mathbb{R}$ is a Lipschitz function if $||f|| < \infty$, where

$$||f|| \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \vee |f(0)|\,.$$

For all x and y in V, we then have

$$|f(x) - f(y)| \leq ||f||\,|x - y|$$

and

$$|f(x)| \leq ||f||(1 + |x|)\,.$$

Scalable Lipschitz Differentiability

We say that $f : V \to \mathbb{R}$ is scalable Lipschitz differentiable at x,

if there exists a function $\wedge \boldsymbol{f_x} : V \to \mathbb{R}$ such that

$$\lim_{|y| \downarrow 0} \frac{|f(x+y) - f(x) - \wedge f_x(y)|}{|y|} = 0,$$

where $\wedge f_x(\cdot)$ is a Lipschitz function, or

$$\| \wedge f_x \| < \infty,$$

and $\wedge f_x(\cdot)$ is scalable, or

$$\wedge f_x(\lambda y) = \lambda \wedge f_x(y)$$

for all $\lambda \geq 0$.

Note: We typically write $\wedge \boldsymbol{f_x(y)}$ as $\wedge f(x; y)$.

Calculus of Scalable Lipschitz Derivatives

1. $f : \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $x \in \mathbb{R}^n$, with Jacobian matrix $Df(x)$. Then
$$\wedge f(x; y) = y \cdot Df(x).$$

2. Addition Formula:
$$\Lambda(f+g)(x;y) = \Lambda f(x;y) + \Lambda g(x;y).$$

3. Multiplication Formula:
$$\Lambda(fg)(x;y) = f(x) \cdot \Lambda g(x;y) + g(x) \cdot \Lambda f(x;y).$$

4. Composition Formula:
$$\Lambda(f \circ g)(x;y) = \Lambda f\left(g(x); g(x) \cdot \Lambda g(x;y)\right).$$

5. Maximum Formula:
$$\begin{aligned} &\Lambda(\max(f,g))(x;y) \\ &\quad = \Lambda f(x;y) \cdot 1_{\{f(x)>g(x)\}} + \Lambda g(x;y) \cdot 1_{\{f(x)<g(x)\}} \\ &\qquad + \max(\Lambda f, \Lambda g)(x;y) \cdot 1_{\{f(x)=g(x)\}}. \end{aligned}$$