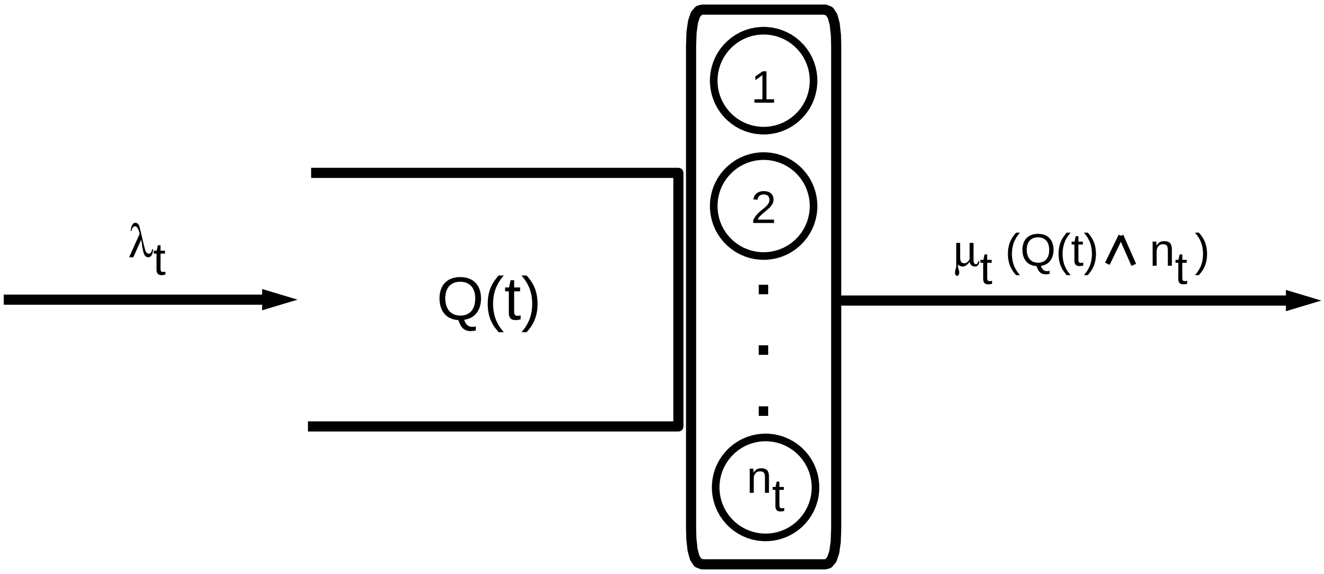
$\mathrm{M_t/M_t/n_t}$: Strong Approximations

Parameters: λ_t, μ_t, n_t.



$Q = \{\, Q(t) \mid t \geq 0 \,\}$: total number in system.

Model:

$$Q(t) \equiv Q(0) + A_1\left(\int_0^t \lambda_s ds\right) - A_2\left(\int_0^t \mu_s \cdot (Q(s) \wedge n_s) ds\right),$$

where A_1 and A_2 are two independent Poisson (1) processes.

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Uniform Acceleration of the $\mathrm{M_t/M_t/n_t}$ Queue

$\lambda \leftrightarrow \eta\lambda, \;\; n \leftrightarrow \eta n$.

Take the limit as $\eta \to \infty$.

Physical interpretation:
scaling up capacity in response to a similar scale up of the offered load.

Formally:
for any $\eta > 0$, consider

$$Q^{\eta}(t) = Q^{\eta}(0) \quad + \quad A_1\left(\int_0^t \eta\lambda_s ds\right)$$
$$\quad - \quad A_2\left(\int_0^t \mu_s \cdot \left(Q^{\eta}(s) \wedge \eta n_s\right) ds\right).$$

$\mathbf{M_t/M_t/n_t}$ Fluid
Limit, Approximation, Model

Assume λ and μ are locally integrable functions. Then

$$\lim_{\eta\to\infty} \frac{1}{\eta}\, Q^{\eta}(t) = Q^{(0)}(t) \quad u.o.c., a.s.$$

given convergence at $t = 0$.

Here

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \left[\lambda_s - \mu_s \cdot \left(Q^{(0)}(s) \wedge n_s\right)\right] ds\,,$$

or more compactly

$$\frac{d}{dt}\, Q^{(0)}(t) = \lambda_t - \mu_t \cdot \left(Q^{(0)}(t) \wedge n_t\right).$$

$M_t/M_t/n_t$ Diffusion Limit

Assume λ and μ are locally integrable functions. Then

$$\lim_{\eta\to\infty} \sqrt{\eta} \left[\frac{1}{\eta} Q^{\eta}(t) - Q^{(0)}(t) \right] \stackrel{d}{=} Q^{(1)}(t) \, ,$$

given convergence at $t = 0$.

The convergence is in $D[0, \infty]$, and

$$\begin{aligned} Q^{(1)}(t) &= Q^{(1)}(0) \\ &\quad - \int_0^t \mu_s \mathbf{1}_{\{Q^{(0)}(s) < n_s\}} Q^{(1)}(s)^+ ds \\ &\quad + \int_0^t \mu_s \mathbf{1}_{\{Q^{(0)}(s) \le n_s\}} Q^{(1)}(s)^- ds \\ &\quad + B\left(\int_0^t \lambda_s + \mu_s \cdot \left(Q^{(0)}(t) \wedge n_s \right) ds \right) \, . \end{aligned}$$

Here $B(\cdot)$ is a standard Brownian motion.

Strong Approximations: $\mathrm{A_i(t)} \leftrightarrow \mathrm{t + B_i(t)}$

$$\frac{1}{\eta} Q^{\eta}(t) \approx \frac{1}{\eta} Q^{\eta}(0) +$$

$$+ \quad \int_0^t \left[\lambda_s - \mu_s \left(\frac{1}{\eta} \; Q^{\eta}(s) \wedge n_s \right) \right] ds$$

$$+ \quad \frac{1}{\eta} B_1 \left(\eta \int_0^t \lambda_s ds \right) - \frac{1}{\eta} B_2 \left(\eta \int_0^t \mu_s \left(\frac{1}{\eta} Q^{\eta}(s) \wedge n_s \right) ds \right)$$

1. FSLLN: As $\eta \uparrow \infty$, $\frac{1}{\eta} Q^{\eta} \to Q^{(0)}$ u.o.c., a.s., given convergence at $t = 0$.

Here $Q^{(0)}$ is the unique solution to the ODE

$$\frac{d}{dt} Q^{(0)}(t) = \lambda_t - \mu_t \left(Q^{(0)}(t) \wedge n_t \right) , \quad t \geq 0.$$

Proof: FSLLN for the B_i's, combined with Gronwall.

2. FCLT: As $\eta \uparrow \infty$, $\sqrt{\eta} \left[\frac{1}{\eta} Q^{\eta} - Q^{(0)} \right] \xrightarrow{d} Q^{(1)}$,

given convergence at $t = 0$.

Here $Q^{(1)}$ is the unique solution to the SDE...

Proof:

Proof of FCLT

1. Brownian Term: as before, based on self-similarity & additivity & time-change, we get that it is distributed as:

$$B\left(\int_0^t [\lambda_s + \mu_s(Q^{(0)}(s) \wedge n_s)]ds\right), \quad t \geq 0.$$

2. Drift $= \sqrt{\eta}\ \int_0^t \left[f_s\left(\frac{1}{\eta}\, Q^\eta(s)\right) - f_s\left(Q^{(0)}(s)\right)\right] ds$

where $\quad f_t(x) = \lambda_t - \mu_t(x \wedge n_t)\,, \quad x \in \mathbb{R}^1.$

If indeed $\quad Q^\eta(t) \stackrel{d}{=} \eta Q^{(0)}(t) + \sqrt{\eta}\, Q^{(1)}(t) + o\left(\sqrt{\eta}\right),$

then letting $\epsilon = 1/\sqrt{\eta}$,

Drift $\stackrel{d}{\approx} \displaystyle\int_0^t \frac{1}{\epsilon}[f_s(Q^{(0)}(s) + \epsilon Q^{(1)}(s)) - f_s(Q^{(0)}(s))]ds$

$\underset{\epsilon\downarrow\infty}{\longrightarrow} \displaystyle\int_0^t \wedge f_s(Q^{(0)}(s); Q^{(1)}(s))ds$

in which $\quad \wedge f_t(x;y) = \text{“}\lim_{\epsilon\downarrow 0}\text{”}\ \frac{1}{\epsilon}\,[f_t(x+\epsilon y) - f_t(x)],$

must be defined for continuous, but non-differentiable functions.

"Extended Calculus"

Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be continuous at x, with left- and right-hand derivatives at x. Then,

$$\wedge f(x; y) = f'(x+)y^+ - f'(x-)y^- \,, \quad x, y \in \mathbb{R}^1.$$

Example:

$$\begin{aligned} f(x) &= x \wedge a \\ \wedge f(x; y) &= y\,1_{x<a} + 1_{x=a}[0.y^+ - 1.y^-] = \\ &= y\,1_{x<a} - y^- 1_{x=a} = y^+ 1_{x<a} - y^- 1_{x \geq a} \end{aligned}$$

Example: Slutsky's Theorem (extended)

Suppose $X_n \to \mu$, $\sqrt{n}\,(X_n - \mu) \xrightarrow{d} Z$, as $n \uparrow \infty$.

Then $f(x_n) \to f(\mu)$ and

$$\sqrt{n}\,[f(X_n) - f(\mu)] \xrightarrow{d} \wedge f(\mu; Z)$$

$$= f'(\mu+)\,Z^+ - f'(\mu^-)\,Z^-$$

$$(= f'(\mu)\,Z \qquad \text{when } \exists\, f'(\mu)).$$

Example: FCLT for $M_t/M_t/n_t$

Recall

$$Q^{(1)}(t) \stackrel{d}{\approx} \int_0^t \wedge f_s(Q^{(0)}(s); Q^{(1)}(s))ds + B(\cdots)$$

where $f_t(x) = \lambda_t - \mu_t(x \wedge n_t)$.

Since $\wedge f_t(x,y) = -\mu_t[y^+ 1_{x<n_t} - y^- 1_{x \le n_t}]$, we conclude

$$\textbf{FCLT:} \quad \lim_{\eta \uparrow \infty} \sqrt{\eta} \left[\frac{1}{\eta} Q^\eta(t) - Q^{(0)}(t) \right] \stackrel{d}{=} Q^{(1)}, \quad t \ge 0,$$

where $Q^{(1)}$ is the unique solution of the following SDE:

$$\begin{aligned} Q^{(1)}(t) = Q^{(1)}(0) \quad &- \int_0^t \mu_s \cdot 1_{\{Q^{(0)}(s)<n_s\}} Q^{(1)}(s)^+ ds \\ &+ \int_0^t \mu_s \cdot 1_{\{Q^{(0)}(s)\le n_s\}} Q^{(1)}(s)^- ds \\ + B\left(\int_0^t [\lambda_s + \mu_s(Q^{(0)}(s) \wedge n_s)]ds\right), &\qquad t \ge 0. \end{aligned}$$

Differential Equations for the Diffusion Moments of the $\mathrm{M_t/M_t/n_t}$ Queue

If $\{\, t \mid Q^{(0)}(t) = n_t \,\}$ has Lebesque measure zero, then $Q^{(1)}(\cdot)$ is a **Gaussian** process. Furthermore,

$$\frac{d}{dt}\, \mathsf{E}\left[Q^{(1)}(t)\right] = -\mu_t 1_{\{Q^{(0)}(t) \le n_t\}} \mathsf{E}\left[Q^{(1)}(t)\right],$$

$$\begin{aligned}\frac{d}{dt}\, \mathsf{Var}\left[Q^{(1)}(t)\right] &= -2\mu_t 1_{\{Q^{(0)}(t) \le n_t\}} \mathsf{Var}\left[Q^{(1)}(t)\right] \\ &\quad + \lambda_t + \mu_t \left(Q^{(0)}(t) \wedge n_t\right),\end{aligned}$$

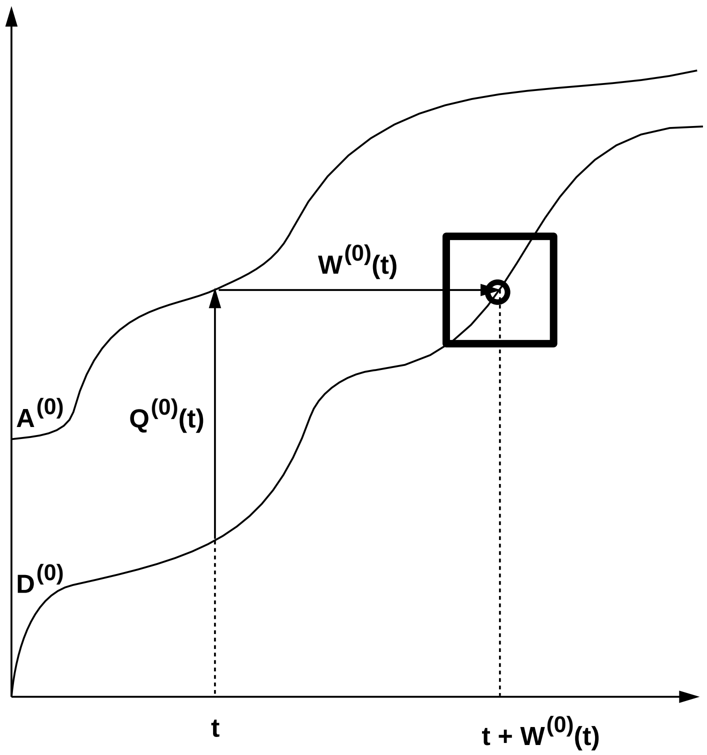
and

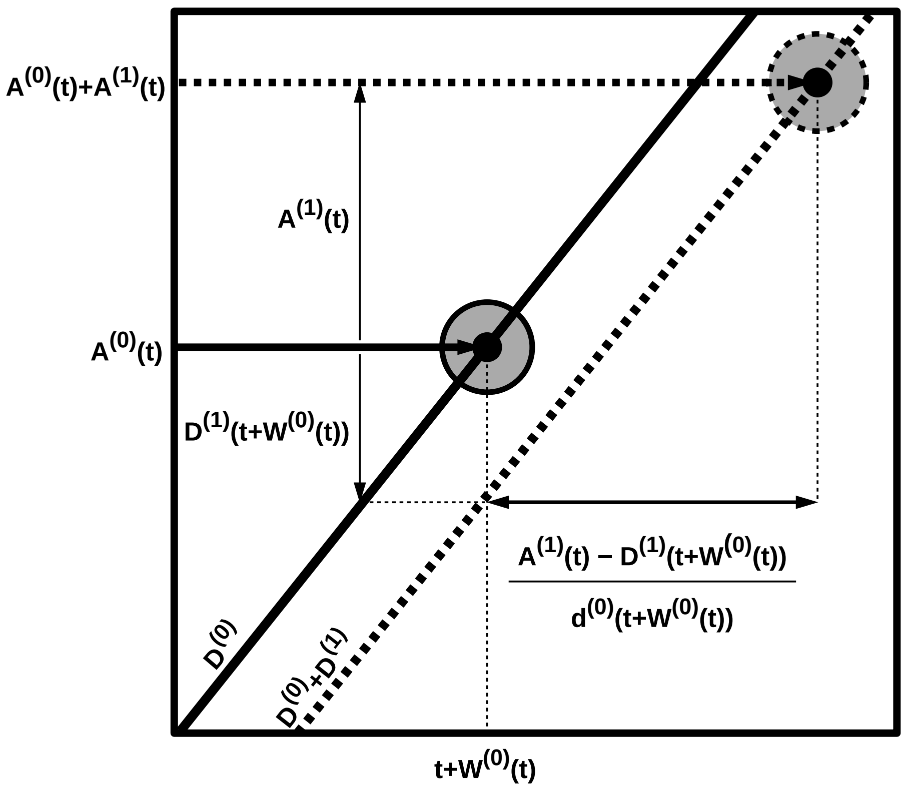
$$\begin{aligned}&\frac{d}{dt}\, \mathsf{Cov}\left[Q^{(1)}(s), Q^{(1)}(t)\right] \\ &\quad = -\mu_t 1_{\{Q^{(0)}(t) \le n_t\}} \mathsf{Cov}\left[Q^{(1)}(s), Q^{(1)}(t)\right].\end{aligned}$$

The above is solvable numerically,

in a spreadsheet, via forward increments.

Waiting Time





Virtual Waiting Times for the $\mathbf{M_t/M_t/n_t/\infty/FCFS}$ Queue

Fix a time τ.

Define $\{\, \hat{Q}(t) \mid t \geq 0 \,\}$ to be the queue length process associated with an $M_t/M_t/n_t$ system, with parameters μ_t and n_t as before, but with arrival rates $\hat{\lambda}_t$ that are modified as follows:

$$\hat{\lambda}_t = \begin{cases} \lambda_t & \text{if } t \leq \tau, \\ 0 & \text{if } t > \tau. \end{cases}$$

The virtual waiting time for a customer arriving at time τ is $W(\tau) - \tau$ where

$$W(\tau) = \inf \left\{\, t \geq \tau \;\middle|\; \hat{Q}(t) \leq n_t - 1 \,\right\}.$$

The uniformly-accelerated version is

$$W^{\eta}(\tau) = \inf \left\{\, t \geq \tau \;\middle|\; \hat{Q}^{\eta}(t) \leq \eta n_t - 1 \,\right\}.$$

Virtual Waiting Time: Fluid Limit

FSLLN $$\lim_{\eta \to \infty} W^{\eta} = W^{(0)} \text{ a.s.}$$

where

$$W^{(0)}(\tau) = \inf \left\{ t \geq \tau \;\middle|\; \hat{Q}^{(0)}(t) \leq n_t \right\}$$

with

$$\hat{Q}^{(0)}(t) = Q^{(0)}(\tau) - \int_{\tau}^{t} \mu_s n_s ds.$$

The analysis is extendable to the

process $\left\{ W^{(0)}(\tau) \mid \tau \geq 0 \right\}$

(from merely the random variable $W^{(0)}(\tau)$).

Virtual Waiting Time: Diffusion Limit

FCLT $$\lim_{\eta\to\infty} \sqrt{\eta}\,(W^{\eta} - W^{(0)}) \stackrel{d}{=} W^{(1)}$$

where

$$W^{(1)}(\tau) = \frac{\hat{Q}^{(1)}(W^{(0)}(\tau))}{\mu_{W^{(0)}(\tau)} n_{W^{(0)}(\tau)}}$$

If $\hat{Q}^{(1)}$ is a Gaussian process, then Var $\left[W^{(1)}(\tau)\right]$ is calculated via

$$\text{Var}\left[\hat{Q}^{(1)}(W^{(0)}(\tau))\right] = \text{Var}\left[Q^{(1)}(\tau)\right] + \int_{\tau}^{W^{(0)}(\tau)} \mu_s n_s ds.$$

The analysis is extendable to the

stochastic process $\left\{ W^{(1)}(\tau) \mid \tau \geq 0 \right\}$

(from merely the random variable $W^{(1)}(\tau)$).